Math. Nachr. **284**, No. 2–3, 317 – 331 (2011) / **DOI** 10.1002/mana.200810009

The composition in multidimensional Triebel–Lizorkin spaces

Madani Moussai*

Department of Mathematics, Lab. Math. Pure and Applied, University of M'Sila, P.O. Box 166, M'Sila 28000, Algeria

Received 11 January 2008, revised 28 August 2008, accepted 7 November 2008
Published online 8 February 2011

Key words Composition operators, Besov spaces, Triebel–Lizorkin spaces
MSC (2010) 46E35, 47H30

We will establish that the composition operator $T_f : g \to f \circ g$ takes the intersections of Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ with a certain space of bounded and continuous functions to $F^s_{p,q}(\mathbb{R}^n)$, under some conditions on the parameters n, s, p and q. Also, a similar partial result corresponding to the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ will be given.

1 Introduction

Let $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ denote the real-valued functions of the Besov space and the Triebel–Lizorkin space, respectively. For a real function f defined on $\mathbb{R}$ which belongs locally to $F^s_{p,q}(\mathbb{R})$ and vanishes at the origin, we will search for an optimal restriction on the parameters n, s, p and q such that the composition operator $T_f : g \to f \circ g$ takes $F^s_{p,q}(\mathbb{R}^n)$ into itself. We recall that some necessary acting conditions are known; in this sense we own the following result (see [3], [16]–[18]):

Theorem 1.1 *Let $1 + (1/p) < s < (n/p)$ and let $f : \mathbb{R} \to \mathbb{R}$. Then T_f takes $B^s_{p,q}(\mathbb{R}^n)$ (or $F^s_{p,q}(\mathbb{R}^n)$) into itself if and only if there exists some constant c such that $f(t) = c\,t$. The same result holds in the limit case $1 + (1/p) = s < (n/p)$, as soon as $q > 1$ in the case of $B^s_{p,q}(\mathbb{R}^n)$, or $p > 1$ in the case of $F^s_{p,q}(\mathbb{R}^n)$.*

We note that the composition operator problem in $B^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ and in $F^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ is not trivial in the sense that the function f need not be linear, see e.g. [6]–[9]. To study composition operators on intersections has a certain history: in Sobolev spaces by Adams and Frazier [1], [2], and in fractional Sobolev spaces by Brezis and Mironescu [10] and by Maz'ya and Shaposhnikova [13]. In this direction, we will consider T_f on the intersections of $F^s_{p,q}(\mathbb{R}^n)$ with a certain space of bounded and continuous functions $\mathcal{K} = \mathcal{K}(s)$, see (1.1) below. Thus our essential contribution in this paper will concern the Triebel–Lizorkin spaces. Also, we will give a similar partial version for the Besov spaces. But before we formulate this we will use the following notation, which depends on the choice of the parameter s; we put:

$$\mathcal{K} := \begin{cases} \bigcap_{0<r<\infty} B^0_{\infty,r}(\mathbb{R}^n) & \textit{if} \quad [s] = 1, \\ W^1_\infty(\mathbb{R}^n) & \textit{otherwise}. \end{cases} \tag{1.1}$$

Theorem 1.2 *Let $1 < p < +\infty$, let $1 \le q \le +\infty$ and let a real number s be such that*

$$s - [s] > \frac{1}{p} \qquad \textit{and} \qquad [s] \ge 1. \tag{1.2}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. Then the composition operator T_f takes $F^s_{p,q}(\mathbb{R}^n) \cap \mathcal{K}$ to $F^s_{p,q}(\mathbb{R}^n)$ if and only if $f(0) = 0$ and $f \in F^{s,\ell oc}_{p,q}(\mathbb{R})$.

* e-mail: mmoussai@yahoo.fr, Phone: +213 3555 65 17, Fax: +213 3555 18 36

Concerning the "philosophy" of this work, we formulate a reasonable conjecture:

Let $1 \le p < \infty$, let $1 \le q \le \infty$ and let $s > 1 + (1/p)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. Then T_f is an operator from $E^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ to $E^s_{p,q}(\mathbb{R}^n)$ if and only if $f(0) = 0$ and $f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$; (here $E^s_{p,q} = B^s_{p,q}$ or $F^s_{p,q}$).

In the case $n = 1$ this conjecture was proved partially for both Triebel–Lizorkin space and Besov space, cf. [8], [9]. Bourdaud in [5] has proved that T_f takes $B^s_{p,q}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ to $B^s_{p,q}(\mathbb{R}^n)$ if $f(0) = 0$ and $f \in B^{t,\ell oc}_{p,\infty}(\mathbb{R})$ for $s > 1$ and $t > s > [s] + (1/p)$. Compared to this result, in Theorem 1.2 we have $f \in F^{s,\ell oc}_{p,q}(\mathbb{R})$, although we are far from the previous conjecture by considering the intersections of $F^s_{p,q}(\mathbb{R}^n)$ with $\mathcal{K}$. Also in the context of the conjecture and still for $1 + (1/p) < s < (n/p)$, we refer to the works of Bourdaud [3], [4] and Runst [16], [17].

In the goal of the brevity of the paper, some remarks are needed for us.

Remark 1.3 The necessity parts in the conjecture as well as in Theorem 1.2 are covered by [17, 5.3.1, Thm. 2, p. 297].

In the context of Remark 1.3, the following conditions are necessary for a function f, such that T_f takes $E^s_{p,q}(\mathbb{R}^n)$ into itself:

(A) $f \in E^{s,\ell oc}_{p,q}(\mathbb{R})$,

(B) f is locally Lipschitz continuous, cf. [4].

We have (A) $\Rightarrow$ (B) if $s > 1 + (1/p)$, using a classical Sobolev embedding. Also if $0 < s < 1$ we have $T_f\big(E^s_{p,q}(\mathbb{R}^n)\big) \subset E^s_{p,q}(\mathbb{R}^n)$ if and only if $f(0) = 0$ and either f is locally Lipschitz continuous (if $E^s_{p,q}(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$) or f is uniformly Lipschitz continuous (if $B^s_{p,q}(\mathbb{R}^n) \not\subset L_\infty(\mathbb{R}^n)$), cf. [4], [17].

Remark 1.4 The conditions $f(0) = 0$ and $f' \in L_\infty(\mathbb{R})$ imply $\| f \circ g \|_p \le \|f'\|_\infty \|g\|_p$ which is sufficient for the estimate of $T_f(g)$ with respect to the $L_p(\mathbb{R}^n)$-norm.

Remark 1.5 For the proof of Theorem 1.2 in the case $n = 1$, we will limit ourselves to a functions "g which is real analytic" in $F^s_{p,q}(\mathbb{R})$ using the ideas of [6, proof of Thm. 7]. The general case, i.e., $g \in F^s_{p,q}(\mathbb{R})$, can be obtained by Fatou's property, cf. [9], [12].

2 Preparations

2.1 Notation

We work in Euclidean spaces $\mathbb{R}^n$ with $n = 1, 2, \ldots$ All distribution spaces are contained in the distribution space $\mathcal{S}'(\mathbb{R}^n)$. All functions are assumed to be real-valued. By $\| - \|_p$ we denote the L_p norm. We denote by $W^1_\infty(\mathbb{R}^n)$ the space of bounded functions such that the first order weak derivatives are bounded, equipped with the norm

$$\|f\|_{W^1_\infty(\mathbb{R}^n)} := \|f\|_\infty + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_\infty .$$

For a space E of tempered distributions defined on $\mathbb{R}^n$ the associated local space is defined by

$$E^{\ell oc} := \{f \in \mathcal{S}' : \varphi f \in E, \forall \varphi \in \mathcal{D}(\mathbb{R}^n)\}.$$

We define the differences for an arbitrary function f by

$$\Delta_h f(x) := f(x+h) - f(x) \qquad (\forall h, x \in \mathbb{R}^n),$$

and $\Delta^M_h f = \Delta_h\big(\Delta^{M-1}_h f\big), \ldots$ If s is a real number then $[s]$ denotes the integer part of s, i.e., the largest integer less than or equal to s.

Throughout this paper we will consider parameters s, m, p, q and M, which are supposed to satisfy $s > 0$, $1 < p < \infty$, $1 \le q \le \infty$, and $m, M \in \mathbb{N} \cup \{0\}$. Furthermore we suppose $m = [s]$ and $m < M$. In Subsection 2.3 below also $p = 1$ and $p = \infty$ are admissible in case of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and $p = 1$ in case of Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$, respectively. Also, we will use a *cut-off* function denoted by ρ_t: we fix $\rho \in \mathcal{D}(\mathbb{R})$, a function such that $\operatorname{supp} \rho \subset [-2, 2]$ and $\rho(x) = 1$ if $x \in [-1, 1]$, and we put $\rho_t(x) = \rho(x/t)$.

As usual, constants $c, c_1, \ldots$ are strictly positive and depend only on the fixed parameters n, s, p and q; their values may vary from line to line.

2.2 The *p*-variation function spaces

A function g is said to be of *bounded* p*-variation* if $\nu_p(g) < +\infty$, where

$$\nu_p(g) := \sup\left\{ \left(\sum_{k=1}^{N} |g(t_k) - g(t_{k-1})|^p \right)^{1/p} : \forall \{t_k\}_{k=0}^{N} \subset \mathbb{R},\, t_0 < t_1 < \cdots < t_N \right\}.$$

By $BV_p^1(\mathbb{R})$ we denote the space of primitives of functions of bounded p-variation, and endow it with the semi-norm

$$\| f \|_{BV_p^1(\mathbb{R})} := \inf \nu_p(g),$$

where the infimum is taken of all functions g such that f is a primitive of g. While defining $BV_p^1(\mathbb{R})$ by the primitives, this space is defined as a space of true functions and not of functions modulo almost everywhere. This small subtlety is rather well explained in [7]. The space $BV_p^1(\mathbb{R})$ is not embedded in L_p, however we have at our disposal the embedding

$$B_{p,1}^{1+(1/p)}(\mathbb{R}) \hookrightarrow BV_p^1(\mathbb{R}), \tag{2.1}$$

which is given by Peetre (cf. [15, p. 112] or [6, Thm. 5]) for homogeneous Besov space and can be easily extended to nonhomogeneous Besov space. Also some properties of $BV_p^1(\mathbb{R})$ can be found in [6], [7], [11].

2.3 Some equivalent norms in $B_{p,q}^{s}(\mathbb{R}^n)$ and $F_{p,q}^{s}(\mathbb{R}^n)$

For the definition and general properties of the Besov spaces and the Triebel–Lizorkin spaces we refer to [15], [17], [19], [20]. Also, all the following assertions are proved in Triebel's book [20, pp. 140–144 and p. 194].

(i) The following two expressions

$$\|f\|_p + \left(\int_{\mathbb{R}^n} |h|^{-sq} \left\|\Delta_h^M f\right\|_p^q \frac{\mathrm{d}h}{|h|^n} \right)^{1/q} \quad \text{and} \quad \|f\|_p + \sum_{j=1}^{n} \left(\int_0^1 t^{-sq} \left\|\Delta_{te_j}^M f\right\|_p^q \frac{\mathrm{d}t}{t} \right)^{1/q}$$

define equivalent norms in $B_{p,q}^s(\mathbb{R}^n)$, where $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$.

(ii) Let $s > \frac{n}{\min(p,q)}$. Then a function f belongs to $F_{p,q}^s(\mathbb{R}^n)$ if:

$$\|f\|_{F_{p,q}^s(\mathbb{R}^n)} := \|f\|_p + \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |h|^{-sq} \left|\Delta_h^M f(x)\right|^q \frac{\mathrm{d}h}{|h|^n} \right)^{p/q} \mathrm{d}x \right)^{1/p} < +\infty.$$

(iii) Let $1 \le u \le \infty$ and

$$s > \max\left(\frac{1}{p} - \frac{1}{u}, \frac{1}{q} - \frac{1}{u} \right). \tag{2.2}$$

Then the following expression

$$\|f\|_p + \left(\int_{\mathbb{R}} \left(\int_0^\infty t^{-sq} \left(\frac{1}{t} \int_{|h|\le t} \left|\Delta_h^M f(x)\right|^u \mathrm{d}h \right)^{q/u} \frac{\mathrm{d}t}{t} \right)^{p/q} \mathrm{d}x \right)^{1/p}$$

defines equivalent norm in $F_{p,q}^s(\mathbb{R})$.

(iv) One can replace $\int_{\mathbb{R}^n} \ldots \,\mathrm{d}h/|h|^n$ in (i) and (ii) by $\int_{|h|\le 1} \ldots \,\mathrm{d}h/|h|^n$; also, $\int_0^\infty \ldots \,\mathrm{d}t/t$ in (iii) can be replaced by $\int_0^1 \ldots \,\mathrm{d}t/t$, because of the part of integral which $|h| > 1$ or $t > 1$ can be estimated by the L_p norm of such function.

(v) For any integer $k \ge [s] - 1$ the following expression

$$\|f\|_p + \left\|f^{(k)}\right\|_{E_{p,q}^{s-k}(\mathbb{R})}$$

defines equivalent norm in $E_{p,q}^s(\mathbb{R})$, $\big($recall $E_{p,q}^s(\mathbb{R}) = B_{p,q}^s(\mathbb{R})$ or $F_{p,q}^s(\mathbb{R})\big)$.

3 The Besov case

3.1 The case $n = 1$

We first deal with the case of one dimensional spaces. We will recall the part of [9, Thm. 2] which is in connection with the condition (1.2).

Theorem 3.1 *Let s a real number such that*

$$s - [s] > \frac{p}{\min(p,q)} + \frac{1}{p} - 1 \qquad \textit{and} \qquad [s] \geq 1. \tag{3.1}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(0) = 0$ and $f \in B^{s,\ell oc}_{p,q}(\mathbb{R})$. Then the composition operator T_f takes the space $B^s_{p,q}(\mathbb{R})$ into itself.

In the sequel of this section we put

$$\alpha := \min(p,q) \qquad \beta := \frac{p}{\alpha} + \frac{1}{p} - 1 \qquad \textit{and} \qquad r := \frac{q(\alpha(s-[s]+1-(1/p))-p)}{\alpha q(s-[s]+1-(1/p))-p}. \tag{3.2}$$

The following proposition is the explicit version of Theorem 3.1, which turns to be an essential tool for multidimensional case, cf. Remark 3.3 below. Namely,

Proposition 3.2 *Suppose* (3.1). *Then there exists a constant $c = c(p,q,s) > 0$, such that the inequality*

$$\begin{aligned} \| f \circ g \|_{B^s_{p,q}(\mathbb{R})} \; &\leq \; c \, \| (f\rho_t)^{(m)} \|_{B^{s-m}_{p,q}(\mathbb{R})} \\ &\times \left(1 + \|g\|^{s-m-\beta}_{B^0_{\infty,r}(\mathbb{R})} \, \|g\|^{\beta-(1/p)}_{B^s_{p,q}(\mathbb{R})}\right) (1 + \|g'\|_\infty)^{m-1} \, \|g\|_{B^s_{p,q}(\mathbb{R})} \end{aligned} \tag{3.3}$$

holds, $\forall f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f \in B^{s,\ell oc}_{p,q}(\mathbb{R})$, and $\forall g \in B^s_{p,q}(\mathbb{R})$, and $\forall t \geq \max(1, \|g\|_\infty)$.

Remark 3.3 By the embeddings $B^{s-1}_{p,q}(\mathbb{R}) \hookrightarrow L_\infty(\mathbb{R})$ and $B^s_{p,q}(\mathbb{R}) \hookrightarrow B^0_{\infty,r}(\mathbb{R})$ and by the inequality

$$\|g'\|_{B^{s-1}_{p,q}(\mathbb{R})} \leq c \, \|g\|_{B^s_{p,q}(\mathbb{R})}$$

the second member of (3.3) can be replaced by

$$c \, \| (f\rho_t)^{(m)} \|_{B^{s-m}_{p,q}(\mathbb{R})} (1 + \|g\|_{B^s_{p,q}(\mathbb{R})})^{s-(1/p)},$$

but for technical reasons concerning the case $n \geq 2$ we prefer to keep it in this form.

Remark 3.4 Observe that (3.1) implies $s > 1 + (1/p)$. This restriction has been used also in [5], [8], [9].

Proof of Theorem 3.1 The result follows from Proposition 3.2 by taking $t \geq \max(1, \|g\|_\infty)$ and by Remark 3.3. □

Then it remains to prove Proposition 3.2. We need first to recall the following result proved in [9], (see also [8, Prop. 3]):

Proposition 3.5 *Let $1 + (1/p) < s < 2$. Then there exists a constant $c = c(p,q,s) > 0$, such that the inequality*

$$\| f \circ g \|_{B^s_{p,q}(\mathbb{R})} \leq c \, \|f'\|_{B^{s-1}_{p,q}(\mathbb{R})} \left(\|g\|_{B^s_{p,q}(\mathbb{R})} + \|g\|^{s-(1/p)}_{BV^1_{\alpha(s-(1/p))}(\mathbb{R})} \right) \tag{3.4}$$

holds, $\forall f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f' \in B^{s-1}_{p,q}(\mathbb{R})$, and $\forall g \in B^s_{p,q}(\mathbb{R}) \cap BV^1_{\alpha(s-(1/p))}(\mathbb{R})$.

Proof of Proposition 3.2 We will prove the assertion by induction on m.

Step 1. The case $m = 1$. We use an inequality of Galiardo–Nirenberg's type, similar to Theorem 2.2.5 of [17]. For all $\theta \in]0,1[$ it holds:

$$\|g\|_{B^{\theta s}_{p/\theta,1}(\mathbb{R})} \; \leq \; c \, \|g\|^{1-\theta}_{B^0_{\infty,r}(\mathbb{R})} \, \|g\|^\theta_{B^s_{p,q}(\mathbb{R})} \qquad \left(\forall g \in B^s_{p,q}(\mathbb{R}), \quad r = \frac{1-\theta}{1-(\theta/q)} \right). \tag{3.5}$$

We choose

$$\theta := \frac{p}{\alpha(s-(1/p))} \qquad (\theta \in]0,1[\quad \text{see condition (3.1)}). \tag{3.6}$$

Then combining (2.1) and (3.5) we obtain the following chain of embeddings

$$B^s_{p,q}(\mathbb{R}) \hookrightarrow B^{\theta s}_{p/\theta,1}(\mathbb{R}) \hookrightarrow B^{1+(\theta/p)}_{p/\theta,1}(\mathbb{R}) \hookrightarrow BV^1_{p/\theta}(\mathbb{R}).$$

Let now $t > \|g\|_\infty$, which implies $\rho_t \circ g = 1$. Using both Proposition 3.5, with $f\rho_t$ instead of f, and (3.5), we get

$$\begin{aligned} \|f \circ g\|_{B^s_{p,q}(\mathbb{R})} &:= \|(f\rho_t) \circ g\|_{B^s_{p,q}(\mathbb{R})} \\ &\leq c\,\|(f\rho_t)'\|_{B^{s-1}_{p,q}(\mathbb{R})} \Big(1 + \|g\|^{s-1-\beta}_{B^0_{\infty,r}(\mathbb{R})} \|g\|^{\beta-(1/p)}_{B^s_{p,q}(\mathbb{R})}\Big) \|g\|_{B^s_{p,q}(\mathbb{R})}. \end{aligned}$$

By taking into account that $\rho_{2t}\rho_t' = \rho_t'$, if $t \geq \max(1, \|g\|_\infty)$, cf. [17, 4.7], then we obtain

$$\begin{aligned} \|\,(f\rho_t)'\,\|_{B^{s-1}_{p,q}(\mathbb{R})} &:= \|f'\rho_t + f\rho_{2t}(\rho_t)'\|_{B^{s-1}_{p,q}(\mathbb{R})} \\ &\leq \|f'\rho_t\|_{B^{s-1}_{p,q}(\mathbb{R})} + \|(\rho_t)'\|_{B^{s-1}_{\infty,q}(\mathbb{R})} \|f\rho_{2t}\|_{B^{s-1}_{p,q}(\mathbb{R})} \\ &\leq c\,\Big(\|f'\rho_t\|_{B^{s-1}_{p,q}(\mathbb{R})} + \|f\rho_{2t}\|_{B^{s-1}_{p,q}(\mathbb{R})}\Big); \end{aligned}$$

the last inequality follows from the properties of pointwise multiplication in Besov spaces, cf. [17, Thm. 4.7.1, p. 229], namely

$$B^{s-1}_{\infty,q}(\mathbb{R}) \cdot B^{s-1}_{p,q}(\mathbb{R}) \hookrightarrow B^{s-1}_{p,q}(\mathbb{R}).$$

To prove (3.5) we first recall that for all sequence $\{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi_0 \subset \{\xi : |\xi| \leq 2\}$, $\operatorname{supp}\varphi_j \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ if $j = 1, 2, \ldots$ and

$$\sum_{j=0}^\infty \varphi_j(\xi) := 1 \qquad (\forall \xi \in \mathbb{R}^n),$$

we have equivalent norm in Besov spaces defines by the formula

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{j=0}^\infty 2^{sjq}\,\|\mathcal{F}^{-1}(\varphi_j\,\mathcal{F}f)\|_p^q\right)^{1/q} < +\infty, \tag{3.7}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the inverse Fourier transform, respectively; see e.g., [17] or [19]. Now since

$$2^{\theta sj}\,\|\,\mathcal{F}^{-1}(\varphi_j\,\mathcal{F}g)\,\|_{p/\theta} \leq \|\,\mathcal{F}^{-1}(\varphi_j\,\mathcal{F}g)\,\|_\infty^{1-\theta}\,\left(2^{sj}\,\|\,\mathcal{F}^{-1}(\varphi_j\,\mathcal{F}g)\,\|_p\right)^\theta,$$

then we sum over the j and we conclude using the Hölder's inequality with $(\theta/q) + ((1-\theta)/r) = 1$.

Step 2. The case $[s] = m+1$. Before applying the induction assumption, we will give some preparations. We put

$$f_1(x) := f(x) - \sum_{j=1}^{m+1} \frac{f^{(j)}(0)}{j\,!}\,x^j \qquad (x \in \mathbb{R}).$$

We have $f_1 \in B^{s,\ell oc}_{p,q}(\mathbb{R})$. Also for all $j \in \{1, \ldots, m+1\}$ we have the four following estimates:

(i)

$$\|g'\|_{B^{s-j}_{p,q}(\mathbb{R})} \leq c\,\|g\|_{B^s_{p,q}(\mathbb{R})},$$

(ii) using the Banach algebra property of $L_\infty(\mathbb{R}) \cap B^s_{p,q}(\mathbb{R})$ cf. [17, Thm. 4.6.4/2, p. 222], i.e.,

$$\|g_1 \cdot g_2\|_{B^s_{p,q}(\mathbb{R})} \le c\left(\|g_1\|_\infty \|g_2\|_{B^s_{p,q}(\mathbb{R})} + \|g_1\|_{B^s_{p,q}(\mathbb{R})}\|g_2\|_\infty\right) \quad \left(\forall g_1, g_2 \in L_\infty(\mathbb{R}) \cap B^s_{p,q}(\mathbb{R})\right),$$

we obtain

$$\begin{aligned}\|g^j\|_{B^s_{p,q}(\mathbb{R})} &\le c_1 \left(\|g\|_\infty \left\|g^{j-1}\right\|_{B^s_{p,q}(\mathbb{R})} + \|g\|_{B^s_{p,q}(\mathbb{R})} \left\|g^{j-1}\right\|_\infty\right) \\ &\le c_2 \|g\|_\infty \left(\|g\|_\infty \left\|g^{j-2}\right\|_{B^s_{p,q}(\mathbb{R})} + \|g\|_{B^s_{p,q}(\mathbb{R})} \|g\|_\infty^{j-2}\right) \\ &\quad + c_1 \|g\|_{B^s_{p,q}(\mathbb{R})} \|g\|_\infty^{j-1} \\ &\cdots \\ &\le c_{m+1} \|g\|_{B^s_{p,q}(\mathbb{R})} \|g\|_\infty^{j-1},\end{aligned}$$

(iii)

$$\left|f^{(j)}(0)\right| := \left|(f\rho_t)^{(j)}(0)\right| \le \left\|(f\rho_t)^{(j)}\right\|_\infty \le c\left\|(f\rho_t)^{(j)}\right\|_{B^{s-j}_{p,q}(\mathbb{R})},$$

(iv)

$$\begin{aligned}\left\| f_1^{(j-1)} \circ g\right\|_p &:= \left\|(f_1\rho_t)^{(j-1)} \circ g\right\|_p \\ &\le \left\|(f_1\rho_t)^{(j)}\right\|_\infty \|g\|_p \le c\left\|(f_1\rho_t)^{(j)}\right\|_{B^{s-j}_{p,q}(\mathbb{R})} \|g\|_{B^s_{p,q}(\mathbb{R})}.\end{aligned}$$

Now since $f_1^{(j)}(0) = 0$ $(j = 1, 2, \ldots, m+1)$ and

$$\beta < s - (m+1) \le 1 \quad \text{(see condition (3.1))},$$

then by Section 2.3/(v), and by the Banach algebra property of $L_\infty(\mathbb{R}) \cap B^{s-j}_{p,q}(\mathbb{R})$ again, we obtain

$$\begin{aligned}\|f \circ g\|_{B^s_{p,q}(\mathbb{R})} &\le \|f_1 \circ g\|_{B^s_{p,q}(\mathbb{R})} + \sum_{j=1}^{m+1} \frac{|f^{(j)}(0)|}{j\,!} \|g^j\|_{B^s_{p,q}(\mathbb{R})} \\ &\le c_1 \Bigg(\|g'\|_\infty \|f_1' \circ g\|_{B^{s-1}_{p,q}(\mathbb{R})} + \|g'\|_{B^{s-1}_{p,q}(\mathbb{R})} \|f_1' \circ g\|_\infty + \|f_1 \circ g\|_p \\ &\quad + \sum_{j=1}^{m+1} \|(f\rho_t)^{(j)}\|_{B^{s-j}_{p,q}(\mathbb{R})} \|g\|_\infty^{j-1} \|g\|_{B^s_{p,q}(\mathbb{R})}\Bigg) \\ &\le c_2 \Big\{\|g'\|_\infty^2 \|f_1'' \circ g\|_{B^{s-2}_{p,q}(\mathbb{R})} + \|g'\|_\infty \Big(\|f_1' \circ g\|_p + \|g'\|_{B^{s-2}_{p,q}(\mathbb{R})} \|f_1'' \circ g\|_\infty\Big) \\ &\quad + \|g'\|_{B^{s-1}_{p,q}(\mathbb{R})} \|f_1'\|_\infty + \|(f_1\rho_t)'\|_{B^{s-1}_{p,q}(\mathbb{R})} \|g\|_{B^s_{p,q}(\mathbb{R})}\Big\} \\ &\quad + c_1 \left(\sum_{j=1}^{m+1} \|(f\rho_t)^{(j)}\|_{B^{s-j}_{p,q}(\mathbb{R})} \|g\|_\infty^{j-1} \|g\|_{B^s_{p,q}(\mathbb{R})}\right) \\ &\cdots \\ &\le c_m \Bigg(\|g'\|_\infty^m \|f_1^{(m)} \circ g\|_{B^{s-m}_{p,q}(\mathbb{R})} + \sum_{j=1}^{m+1} \|(f\rho_t)^{(j)}\|_{B^{s-j}_{p,q}(\mathbb{R})} \|g\|^j_{B^s_{p,q}(\mathbb{R})} \\ &\quad + \sum_{j=1}^{m+1} \left\|(f_1\rho_t)^{(j-1)}\right\|_{B^{s-j}_{p,q}(\mathbb{R})} \|g\|_{B^s_{p,q}(\mathbb{R})} \\ &\quad + \sum_{j=1}^{m+1} \left\|(f_1\rho_t)^{(j)}\right\|_{B^{s-1}_{p,q}(\mathbb{R})} \|g'\|_\infty^{j-1} \|g'\|_{B^{s-j}_{p,q}(\mathbb{R})}\Bigg).\end{aligned}$$

The estimate of the three last terms is obvious, however by the assumption of induction on m applied to $f_1^{(m)}$ it follows

$$\| f_1^{(m)} \circ g \|_{B^{s-m}_{p,q}(\mathbb{R})} \le c_{m+1} \| (f_1 \rho_t)^{(m+1)} \|_{B^{s-(m+1)}_{p,q}(\mathbb{R})} \times \Big(1 + \|g\|^{s-(m+1)-\beta}_{B^0_{\infty,r}(\mathbb{R})} \|g\|^{\beta-(1/p)}_{B^{s-m}_{p,q}(\mathbb{R})}\Big) \|g\|_{B^{s-m}_{p,q}(\mathbb{R})}.$$

We note that it is clear in this case

$$\theta := \frac{p}{\alpha(s-m+1-(1/p))} \qquad \text{(see (3.6))}. \tag{3.8}$$

□

3.2 The case $n = 2, 3, \ldots$

We turn now to the multidimensional case, so we prove the following precise result:

Theorem 3.6 *Let s and f be as in Theorem* 3.1. *Suppose $p \le q$. Then the composition operator T_f takes $B^s_{p,p}(\mathbb{R}^n) \cap \mathcal{K}$ to $B^s_{p,q}(\mathbb{R}^n)$.*

P r o o f. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$ we put

$$\widehat{x}_j := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \quad \textit{and} \quad g_{\widehat{x}_j}(y) := g(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n). \tag{3.9}$$

By the inequality of Minkowski with respect to $L_{q/p}$, it follows

$$\begin{aligned}
&\int_0^1 \Big(t^{-sp} \int_{\mathbb{R}^n} \big| \Delta^M_{te_j}(f\circ g)(x) \big|^p \,\mathrm{d}x \Big)^{q/p} \frac{\mathrm{d}t}{t} \\
&:= \int_0^1 \Big(t^{-sp} \int_{\mathbb{R}^{n-1}} \big\| \Delta^M_t (f \circ g_{\widehat{x}_j}) \big\|_p^p \,\mathrm{d}\widehat{x}_j \Big)^{q/p} \frac{\mathrm{d}t}{t} \\
&\le \Bigg(\int_{\mathbb{R}^{n-1}} \Big(\int_0^1 \Big(t^{-s} \big\| \Delta^M_t (f\circ g_{\widehat{x}_j}) \big\|_p \Big)^q \frac{\mathrm{d}t}{t} \Big)^{p/q} \mathrm{d}\widehat{x}_j \Bigg)^{q/p} \\
&:= \Big(\int_{\mathbb{R}^{n-1}} \big\| f \circ g_{\widehat{x}_j} \big\|^p_{B^s_{p,q}(\mathbb{R})} \,\mathrm{d}\widehat{x}_j \Big)^{q/p}.
\end{aligned}$$

Also from Proposition 3.2 and the embedding $B^s_{p,p}(\mathbb{R}) \hookrightarrow B^s_{p,q}(\mathbb{R})$, it holds

$$\big\| f \circ g_{\widehat{x}_j} \big\|_{B^s_{p,q}(\mathbb{R})} \le c \big\| (f\rho_t)^{(m)} \big\|_{B^{s-m}_{p,q}(\mathbb{R})} \times \Big(1 + \big\| g'_{\widehat{x}_j} \big\|_\infty \Big)^{m-1} \Big(1 + \big\| g_{\widehat{x}_j} \big\|^{s-m-(1/p)}_{B^0_{\infty,r}(\mathbb{R})} \Big) \big\| g_{\widehat{x}_j} \big\|_{B^s_{p,p}(\mathbb{R})},$$

where r is given by (3.2), and for all $t \ge \max(1, \|g\|_\infty)$. Now the following inequality

$$\Big(\int_{\mathbb{R}^{n-1}} \|g_{\widehat{x}_j}\|^p_{B^s_{p,p}(\mathbb{R})} \mathrm{d}\widehat{x}_j \Big)^{1/p} \le c\, \|g\|_{B^s_{p,p}(\mathbb{R}^n)}$$

yields

$$\| f \circ g \|_{B^s_{p,q}(\mathbb{R}^n)} \le c \,\|(f\rho_t)^{(m)}\|_{B^{s-m}_{p,q}(\mathbb{R})} \,(1 + \|g\|_{\mathcal{K}})^{s-1-(1/p)} \,\|g\|_{B^s_{p,p}(\mathbb{R}^n)}. \tag{3.10}$$

This completes the proof. □

We can consider the case $p > q$ by the following result.

Corollary 3.7 *Let s and f be as in Theorem* 3.1. *Let ε be a real number such that*

$$0 \le \varepsilon < s, \quad \textit{and} \quad \varepsilon \ne 0 \quad \textit{if} \quad p > q.$$

Then the composition operator T_f takes $B^s_{p,p}(\mathbb{R}^n) \cap \mathcal{K}$ to $B^{s-\varepsilon}_{p,q}(\mathbb{R}^n)$.

Proof. The case $p \le q$ is given by the previous theorem. Assume that $p > q$. Since $B^{s,\ell oc}_{p,q}(\mathbb{R}) \hookrightarrow B^{s,\ell oc}_{p,\infty}(\mathbb{R})$, then Theorem 3.6 implies that T_f takes $B^s_{p,p}(\mathbb{R}^n) \cap \mathcal{K}$ to $B^s_{p,\infty}(\mathbb{R}^n)$. Then we deduce the result by the embedding $B^s_{p,\infty}(\mathbb{R}^n) \hookrightarrow B^{s-\varepsilon}_{p,q}(\mathbb{R}^n)$. □

4 Proof of Theorem 1.2

The proof is based on an inequality of type (3.4) for the Triebel–Lizorkin spaces, which is an essential part of this section.

Proposition 4.1 *Let $1 + (1/p) < s < 2$. Then there exists a constant $c = c(p,q,s) > 0$, such that the inequality*

$$\| f \circ g \|_{F^s_{p,q}(\mathbb{R})} \le c \, \|f'\|_{F^{s-1}_{p,q}(\mathbb{R})} \left(\|g\|_{F^s_{p,q}(\mathbb{R})} + \|g\|^{s-(1/p)}_{BV^1_{sp-1}(\mathbb{R})} \right) \tag{4.1}$$

holds, $\forall f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f' \in F^{s-1}_{p,q}(\mathbb{R})$, and $\forall g \in F^s_{p,q}(\mathbb{R})$.

We turn to the proof of Theorem 1.2. Let f be as in Theorem 1.2, and let $g \in F^s_{p,q}(\mathbb{R}^n) \cap \mathcal{K}$.

Step 1. The case $n = 1$. Since $F^s_{p,q}(\mathbb{R}) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R})$, then (3.5) (with $\max(p,q)$ instead of q) leads to

$$\|g\|_{B^{\theta s}_{p/\theta,1}(\mathbb{R})} \le c \, \|g\|^{1-\theta}_{B^0_{\infty,r}(\mathbb{R})} \, \|g\|^{\theta}_{F^s_{p,q}(\mathbb{R})} \qquad \left(\forall g \in F^s_{p,q}(\mathbb{R}), \; r = \frac{1-\theta}{1 - (\theta/\max(p,q))} \right). \tag{4.2}$$

Also we have

$$F^s_{p,q}(\mathbb{R}) \hookrightarrow B^s_{p,\infty}(\mathbb{R}) \hookrightarrow B^{\theta s}_{p/\theta,1}(\mathbb{R}) \hookrightarrow B^{1+(1/(sp-1))}_{sp-1,1}(\mathbb{R}) \hookrightarrow BV^1_{sp-1}(\mathbb{R}). \tag{4.3}$$

Using the induction on m: as in Step 1 and 2 of the proof of Proposition 3.2, for all $t \ge \max(1, \|g\|_\infty)$, we obtain by combining (4.1), and both (4.2) and (4.3) with

$$\theta := \frac{1}{s - m + 1 - (1/p)} \quad \ge \quad \frac{1}{s - (1/p)} \qquad \text{(see (3.8))}, \tag{4.4}$$

an inequality similar to (3.3). Namely:

$$\begin{aligned} \| f \circ g \|_{F^s_{p,q}(\mathbb{R})} &:= \| (f\rho_t) \circ g \|_{F^s_{p,q}(\mathbb{R})} \\ &\le c \, \|(f\rho_t)^m\|_{F^{s-m}_{p,q}(\mathbb{R})} \big(1 + \|g'\|_\infty\big)^{m-1} \big(1 + \|g\|^{s-m-(1/p)}_{B^0_{\infty,r}(\mathbb{R})}\big) \, \|g\|_{F^s_{p,q}(\mathbb{R})}. \end{aligned}$$

Noticing that, as in (3.10), we have the estimate

$$\| f \circ g \|_{F^s_{p,q}(\mathbb{R})} \le c \, \|(f\rho_t)^m\|_{F^{s-m}_{p,q}(\mathbb{R})} \big(1 + \|g\|_{\mathcal{K}}\big)^{s-1-(1/p)} \, \|g\|_{F^s_{p,q}(\mathbb{R})}. \tag{4.5}$$

Step 2. The case $n \ge 2$. Using the notation $g_{\widehat{x}_j}$ of (3.9), and applying the Fubini property of Triebel–Lizorkin spaces (cf. [17, Thm. 2.3.4/2, p. 70]), and (4.5), then we have

$$\begin{aligned} \| f \circ g \|_{F^s_{p,q}(\mathbb{R}^n)} &\le c_1 \sum_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} \|f \circ g_{\widehat{x}_j}\|^p_{F^s_{p,q}(\mathbb{R})} \, \mathrm{d}\widehat{x}_j \right)^{1/p} \\ &\le c_2 \, \|(f\rho_t)^{(m)}\|_{F^{s-m}_{p,q}(\mathbb{R})} \, (1 + \|g\|_{\mathcal{K}})^{s-1-(1/p)} \\ &\quad \times \sum_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} \|g_{\widehat{x}_j}\|^p_{F^s_{p,q}(\mathbb{R})} \, \mathrm{d}\widehat{x}_j \right)^{1/p}. \end{aligned} \tag{4.6}$$

Now by the Fubini property again the last expression in (4.6) is bounded by

$$c_3 \, \big\|(f\rho_t)^{(m)}\big\|_{F^{s-m}_{p,q}(\mathbb{R})} \big(1 + \|g\|_{\mathcal{K}}\big)^{s-1-(1/p)} \, \|g\|_{F^s_{p,q}(\mathbb{R}^n)}.$$

This completes the proof. □

Proof of Proposition 4.1 We first remark that in (4.1) it suffices to take $g \in F^s_{p,q}(\mathbb{R})$, because we have the embedding $F^s_{p,q}(\mathbb{R}) \hookrightarrow BV^1_{sp-1}(\mathbb{R})$, see (4.3). Then we consider a function $g \in F^s_{p,q}(\mathbb{R})$ real analytic (see Remark 1.5). Since

$$\|g'\|_{F^{s-1}_{p,q}(\mathbb{R})} \;\le\; c\,\|g\|_{F^s_{p,q}(\mathbb{R})},$$

the decomposition

$$\Delta_h((f'\circ g)\cdot g')(x) \;:=\; (f'\circ g)(x+h)\,\Delta_h g'(x) \;+\; g'(x)\,\Delta_h(f'\circ g)(x),$$

together with Remark 1.4 lead to

$$\begin{aligned}\|\,f\circ g\,\|_{F^s_{p,q}(\mathbb{R})} &\le \|f\circ g\|_p \;+\; \|f'\|_\infty\,\|g'\|_{F^{s-1}_{p,q}(\mathbb{R})} \;+\; V(f;g)\\ &\le c\,\|f'\|_\infty\,\|g\|_{F^s_{p,q}(\mathbb{R})} \;+\; V(f;g),\end{aligned}$$

where

$$V(f;g) := \left(\int_{\mathbb{R}}\left(\int_0^\infty t^{-(s-1)q}\left(t^{-1}\int_{-t}^{t}|\Delta_h(f'\circ g)(x)|^u\,|g'(x)|^u\,\mathrm{d}h\right)^{q/u}\frac{\mathrm{d}t}{t}\right)^{p/q}\mathrm{d}x\right)^{1/p},$$

and we will estimate it: In the integral with respect to h we restrict ourselves to the interval $[0,t]$ denoting the corresponding expression by $V_+(f;g)$ and noting that the estimate with respect to $[-t,0]$ will be completely similar. Now, since $s-1>(1/p)$ it will be enough to choose a parameter $u>1$ such that

$$\frac{1}{q} \;<\; s-1+\frac{1}{u} \qquad \text{(see condition (2.2))}. \tag{4.7}$$

By means of the elementary inequality

$$|g'(x)| \;\le\; |\Delta_h g'(x)| \;+\; \min(|g'(x)|,\,|g'(x+h)|)$$

we split $V_+(f;g)$ into two parts:

$$\begin{aligned}V_1(f;g) \;:=\; &\left(\int_{\mathbb{R}}\left(\int_0^\infty t^{-(s-1)q}\left(t^{-1}\int_0^t\right.\right.\right.\\ &\left.\left.\left.\times\;|\Delta_h(f'\circ g)(x)|^u\,|\Delta_h g'(x)|^u\,\mathrm{d}h\right)^{q/u}\frac{\mathrm{d}t}{t}\right)^{p/q}\mathrm{d}x\right)^{1/p}, \qquad \textit{and}\\ V_2(f;g) \;:=\; &\left(\int_{\mathbb{R}}\left(\int_0^\infty t^{-(s-1)q}\left(t^{-1}\int_0^t\right.\right.\right.\\ &\left.\left.\left.\times\;|\Delta_h(f'\circ g)(x)|^u\,(\min(|g'(x)|,\,|g'(x+h)|))^u\,\mathrm{d}h\right)^{q/u}\frac{\mathrm{d}t}{t}\right)^{p/q}\mathrm{d}x\right)^{1/p}.\end{aligned}$$

Since $f'\in L_\infty(\mathbb{R})$ the estimate of $V_1(f;g)$ is obvious. For $V_2(f;g)$ we will distinguish two cases:

The case 1. If g' does not vanish on $\mathbb{R}$, then g is a diffeomorphism from $\mathbb{R}$ to itself. By the change of variable $y=g(x)$, and by the inequality

$$\min(|g'(x)|,\,|g'(x+h)|) \;\le\; |g'(x)|^\alpha\,|g'(x+h)|^{1-\alpha} \qquad (0\le\alpha\le 1), \tag{4.8}$$

we have

$$\begin{aligned}V_2(f;g)^p \;\le\; &\left(\sup_{\mathbb{R}}|g'|^{\alpha p-1}\right)\int_{\mathbb{R}}\left(\int_0^\infty t^{-(s-1)q}\right.\\ &\left.\times\left(t^{-1}\int_0^t|\,f'\big(g(g^{-1}(y)+h)\big)-f'(y)\,|^u\,|g'(g^{-1}(y)+h)|^{(1-\alpha)u}\,\mathrm{d}h\right)^{q/u}\frac{\mathrm{d}t}{t}\right)^{p/q}\mathrm{d}y\,;\end{aligned}$$

at this time we choose

$$\alpha > \frac{1}{p}. \tag{4.9}$$

We continue with the next change of variable

$$\Theta := \Theta(h) := g(g^{-1}(y) + h) - y,$$

which satisfies $|\Theta| \leq t \sup_{\mathbb{R}} |g'|$. Choose α such that

$$(1-\alpha)u := 1. \tag{4.10}$$

All this lead to

$$\begin{aligned} V_2(f;g)^p &\leq \Big(\sup_{\mathbb{R}} |g'|^{\alpha p - 1}\Big) \int_{\mathbb{R}} \bigg(\int_0^\infty t^{-(s-1)q} \\ &\quad \times \bigg(t^{-1} \int_{|\Theta| \leq t \sup_{\mathbb{R}} |g'|} |\, f'(y+\Theta) - f'(y)\,|^u \,\mathrm{d}\Theta\bigg)^{q/u} \frac{\mathrm{d}t}{t}\bigg)^{p/q} \mathrm{d}y \\ &\leq \Big(\sup_{\mathbb{R}} |g'|^{\alpha p - 1 + (p/u) + (s-1)p}\Big) \int_{\mathbb{R}} \bigg(\int_0^\infty v^{-(s-1)q} \\ &\quad \times \bigg(v^{-1} \int_{|\Theta| \leq v} |\, f'(y+\Theta) - f'(y)\,|^u \,\mathrm{d}\Theta\bigg)^{q/u} \frac{\mathrm{d}v}{v}\bigg)^{p/q} \mathrm{d}y \\ &\leq c \|f'\|^p_{F^{s-1}_{p,q}(\mathbb{R})} \sup_{\mathbb{R}} |g'|^{sp-1} \qquad \text{(see (4.12) below).} \end{aligned}$$

Also, for the satisfaction of the assertions (4.7), (4.9) and (4.10) we need to find number u, such that

$$\frac{1}{q} + 1 - s < \frac{1}{u} < 1 - \frac{1}{p}. \tag{4.11}$$

The case 2. For the general case, we will need to decompose the integral with respect to x as the following (cf. [6]): Let $\{I_l\}_l$ a family of nonempty open disjoint intervals defined such that the complement of $\bigcup_l I_l$ in $\mathbb{R}$ is the discrete set $\{x \in \mathbb{R} :\ g'(x) = 0\}$. For all l and all $x \in I_l$ we introduce the positive number

$$\eta_l(x) := \operatorname{dist}(x, \text{the right endpoint of } I_l)$$

(possibly $+\infty$ if the endpoint to the right is $+\infty$). Then we have

$$\begin{aligned} V_2(f;g) &:= \bigg(\sum_l \int_{I_l} \bigg(\int_0^\infty t^{-(s-1)q} \Big(t^{-1}\int_0^t \ldots \,\mathrm{d}h\Big)^{q/u} \frac{\mathrm{d}t}{t}\bigg)^{p/q} \mathrm{d}x\bigg)^{1/p} \\ &\leq \bigg(\sum_l \int_{I_l} \Big(\int_0^{\eta_l(x)} \ldots\Big)\bigg)^{1/p} + \bigg(\sum_l \int_{I_l} \Big(\int_{\eta_l(x)}^\infty \ldots\Big)\bigg)^{1/p}. \end{aligned}$$

We denote by

$$V_3(f;g) := \bigg(\sum_l \int_{I_l} \Big(\int_0^{\eta_l(x)} \ldots\Big)\bigg)^{1/p},$$

and by $V_4(f;g)$ the corresponding expression to the integral with respect to $t \geq \eta_l(x)$.

Estimate of $V_3(f;g)$. Both (4.8) and (4.10) yield

$$V_3(f;g) := \left(\sum_l \int_{I_l} |g'(x)|^{\alpha p} \left(\int_0^{\eta_l(x)} t^{-(s-1)q} \times \left(t^{-1}\int_0^t |\Delta_h(f'\circ g)(x)|^u\, |g'(x+h)|\,\mathrm{d}h\right)^{q/u} \frac{\mathrm{d}t}{t}\right)^{p/q} \mathrm{d}x\right)^{1/p}.$$

Now we choose both α and u similar to (4.9) and (4.11), respectively. Also, the fact that the function g_l (the restriction of g to I_l) is a diffeomorphism of I_l onto $g(I_l)$, then, as in the case 1, we will reason with the same changes of variables:

$$y := g_l(x) \quad \textit{for} \quad x \in I_l, \quad \Theta := \Theta(h) := g\big(g_l^{-1}(y)+h\big) - y \quad \textit{with} \quad |\Theta| \le t \sup_{I_l} |g'|.$$

We arrive at

$$\begin{aligned} V_3(f;g)^p \;&\le\; \sum_l \sup_{I_l} |g'|^{\alpha p-1} \int_{g(I_l)} \left(\int_0^{\eta_l(g_l^{-1}(y))} t^{-(s-1)q} \times \left(t^{-1}\int_{|\Theta|\le t \sup_{I_l}|g'|} |\, f'(y+\Theta) - f'(y)\,|^u\, \mathrm{d}\Theta\right)^{q/u} \frac{\mathrm{d}t}{t}\right)^{p/q} \mathrm{d}y \\ &\le \left(\sum_l \sup_{I_l} |g'|^{\alpha p - 1 + (p/u) + (s-1)p}\right) \int_{-\infty}^{\infty} \left(\int_0^{\infty} v^{-(s-1)q} \times \left(v^{-1}\int_{|\Theta|\le v} |\, f'(y+\Theta) - f'(y)\,|^u\, \mathrm{d}\Theta\right)^{q/u} \frac{\mathrm{d}v}{v}\right)^{p/q} \mathrm{d}y \\ &\le\; c\left(\sum_l \sup_{I_l} |g'|^{sp-1}\right) \|f'\|^p_{F^{s-1}_{p,q}(\mathbb{R})}. \end{aligned}$$

Now we claim that

$$\left(\sum_l \sup_{t\in I_l} |g'(t)|^{sp-1}\right)^{1/(sp-1)} \;\le\; c\,\|g\|_{BV^1_{sp-1}}. \tag{4.12}$$

Indeed, since $F^s_{p,q}(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$, then for any I_l there exists $\xi_l \in I_l$ such that

$$|g'(\xi_l)| \;:=\; \sup_{t\in I_l} |g'(t)|.$$

Furthermore we have $g'(\xi_l + \eta_l(\xi_l)) = 0$, and the open intervals $\{\,]\xi_l, \xi_l + \eta_l(\xi_l)[\,\}_l$ are pairwise disjoint. Then the assertion follows from

$$\sum_l \sup_{t\in I_l} |g'(t)|^{sp-1} \;:=\; \sum_l |g'(\xi_l) - g'(\xi_l + \eta_l(\xi_l))|^{sp-1} \;\le\; (\nu_{sp-1}(g'))^{sp-1}.$$

See also [6, proof of Thm. 7].

Estimate of $V_4(f;g)$. By (4.8) with $\alpha = 1$, and by the trivial inequality $|\Delta_h(f' \circ g)(x)| \le 2\,\|f'\|_\infty$, we obtain

$$\begin{aligned} V_4(f;g) \;&\le\; \Bigg(\sum_l \int_{I_l} |g'(x)|^p \,\Bigg(\int_{\eta_l(x)}^\infty t^{-(s-1)q} \\ &\qquad \times \left(t^{-1}\int_0^t |\Delta_h(f'\circ g)(x)|^u \,\mathrm{d}h\right)^{q/u} \frac{\mathrm{d}t}{t}\Bigg)^{p/q} \mathrm{d}x\Bigg)^{1/p} \\ &\le\; c_1\,\|f'\|_\infty \left(\sum_l \int_{I_l} \left(\int_{\eta_l(x)}^\infty t^{-(s-1)q}\,\frac{\mathrm{d}t}{t}\right)^{p/q} |g'(x)|^p\,\mathrm{d}x\right)^{1/p} \\ &\le\; c_2\,\|f'\|_\infty \left(\sum_l \int_{I_l} \eta_l(x)^{-(s-1)p}\,|g'(x)|^p\,\mathrm{d}x\right)^{1/p}. \end{aligned}$$

Again, since g' vanishes at the endpoints of I_l we conclude that

$$\frac{|g'(x)|}{\eta_l(x)^{s-1}} \;:=\; \frac{|g'(x) - g'(x+\eta_l(x))|}{\eta_l(x)^{s-1}} \;\le\; \sup_{h\in\mathbb{R}} \frac{|\Delta_h g'(x)|}{|h|^{s-1}} \qquad (\forall x \in I_l).$$

Thus, the fact that $(1/p) < s-1 < 1$ we can use the norm defined in Section 2.3/(ii), then

$$\begin{aligned} V_4(f;g) \;&\le\; c_1\,\|f'\|_\infty \left(\int_{\mathbb{R}} \left(\sup_{h\in\mathbb{R}} \frac{|\Delta_h g'(x)|}{|h|^{s-1}}\right)^p \mathrm{d}x\right)^{1/p} \\ &\le\; c_2\,\|f'\|_\infty\,\|g'\|_{F^{s-1}_{p,\infty}(\mathbb{R})} \\ &\le\; c_3\,\|f'\|_\infty\,\|g\|_{F^{s}_{p,\infty}(\mathbb{R})}, \end{aligned}$$

and we have the desired result by the embedding $F^s_{p,q}(\mathbb{R}) \hookrightarrow F^s_{p,\infty}(\mathbb{R})$.

Hence, $V_+(f;g)$ can be estimated from above by the right-hand side of (4.1) with a constant c independent of f and g. □

5 Composition between $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$

We will extend our investigation to the boundedness of the composition operator T_f between Besov spaces and Triebel–Lizorkin spaces. We put

$$\mathcal{H} := \begin{cases} L_\infty(\mathbb{R}^n) & \textit{if} \quad [s] = 1, \\ W^1_\infty(\mathbb{R}^n) & \textit{otherwise}. \end{cases}$$

Theorem 5.1 *Let s and f be as in Theorem* 1.2. *Let θ be as in* (4.4). *Then the composition operator T_f takes $B^s_{p,\theta}(\mathbb{R}^n) \cap \mathcal{H}$ to $F^s_{p,q}(\mathbb{R}^n)$.*

We propose to show the following result, more precise than Theorem 5.1, and which is a countrepart of Proposition 4.1 in multidimensional case.

Proposition 5.2 *Let θ be as in* (4.4). *Suppose* (1.2). *Then there exists a constant $c = c(n,s,p,q) > 0$, such that the inequality*

$$\|f\circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \;\le\; c\,\|f'\|_{F^{s-1}_{p,q}(\mathbb{R})} \big(1 + \|g\|_{\mathcal{H}}\big)^{s-1-(1/p)} \|g\|_{B^s_{p,\theta}(\mathbb{R}^n)} \tag{5.1}$$

holds, $\forall f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f' \in F^{s-1}_{p,q}(\mathbb{R})$, and $\forall g \in B^s_{p,\theta}(\mathbb{R}^n) \cap \mathcal{H}$.

P r o o f. *Step 1. The case* $m = 1$. We will use the notation $g_{\widehat{x}_j}$ of (3.9). The Fubini property, and the inequality (4.1) and the embedding $B^{\theta s}_{p/\theta,1}(\mathbb{R}) \hookrightarrow BV^1_{sp-1}(\mathbb{R})$ yield

$$\begin{aligned}
\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} &\leq c_1 \sum_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} \|f \circ g_{\widehat{x}_j}\|^p_{F^s_{p,q}(\mathbb{R})} \, \mathrm{d}\widehat{x}_j \right)^{1/p} \\
&\leq c_2 \, \|f'\|_{F^{s-1}_{p,q}(\mathbb{R})} \sum_{j=1}^n \left(\left(\int_{\mathbb{R}^{n-1}} \|g_{\widehat{x}_j}\|^p_{F^s_{p,q}(\mathbb{R})} \right)^{1/p} \right. \\
&\qquad \left. + \left(\int_{\mathbb{R}^{n-1}} \|g_{\widehat{x}_j}\|^{sp-1}_{B^{\theta s}_{p/\theta,1}(\mathbb{R})} \mathrm{d}\widehat{x}_j \right)^{1/p} \right) \\
&\leq c_3 \, \|f'\|_{F^{s-1}_{p,q}(\mathbb{R})} \Big(\|g\|_{F^s_{p,q}(\mathbb{R}^n)} + \|g\|^{s-(1/p)}_{B^{\theta s}_{p/\theta,1}(\mathbb{R}^n)} \Big);
\end{aligned}$$

where the estimate

$$\left(\int_{\mathbb{R}^{n-1}} \|g_{\widehat{x}_j}\|^{sp-1}_{B^{\theta s}_{p/\theta,1}(\mathbb{R})} \, \mathrm{d}\widehat{x}_j \right)^{1/p} \leq \|g\|^{s-(1/p)}_{B^{\theta s}_{p/\theta,1}(\mathbb{R}^n)}$$

is obtained by the Minkowski's inequality with respect to L_{sp-1}. Now if $g \in B^s_{p,\theta}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$, then we will obtain (5.1) by both the embedding $B^s_{p,\theta}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n)$ and the inequality

$$\|g\|_{B^{\theta s}_{p/\theta,1}(\mathbb{R}^n)} \leq c \, \|g\|^{1-\theta}_\infty \, \|g\|^\theta_{B^s_{p,\theta}(\mathbb{R}^n)} \qquad \text{(see inequality (3.5)).}$$

Step 2. The case $m \geq 2$. We will use induction on m. Hence we have to prove (5.1) with $[s] = m + 1$. Indeed, consider first the function

$$f_1(x) := f(x) - f'(0)x.$$

We have $f_1(0) = f_1'(0) = 0$, $f_1'' \in F^{s-2}_{p,q}(\mathbb{R})$ and $m + 1 + (1/p) < s < m + 2$. Then by the property defines in Section 2.3/(v), and by the induction assumption and by the Banach algebra property of $F^{s-1}_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ (see e.g., [17, Thm. 4.6.4/1, p. 222]), we obtain

$$\begin{aligned}
&\|f_1 \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \\
&\quad \leq c_1 \left(\sum_{\nu=1}^n \|\partial_\nu (f_1 \circ g)\|_{F^{s-1}_{p,q}(\mathbb{R}^n)} + \|f_1 \circ g\|_p \right) \\
&\quad \leq c_2 \left(\|f_1'\|_\infty \|g\|_p + \sum_{\nu=1}^n \Big(\|\partial_\nu g\|_\infty \|f_1' \circ g\|_{F^{s-1}_{p,q}(\mathbb{R}^n)} + \|\partial_\nu g\|_{F^{s-1}_{p,q}(\mathbb{R}^n)} \|f_1'\|_\infty \Big) \right) \\
&\quad \leq c_3 \, \Big(\|f_1'\|_\infty \, \|g\|_{F^s_{p,q}(\mathbb{R}^n)} + \|f_1''\|_{F^{s-2}_{p,q}(\mathbb{R})} \big(1 + \|g\|_\infty\big)^{s-(1/p)-2} \|g\|_{B^{s-1}_{p,\theta}(\mathbb{R}^n)} \|\nabla g\|_\infty \Big).
\end{aligned}$$

Using both the embedding $B^s_{p,\theta}(\mathbb{R}^n) \hookrightarrow B^{s-1}_{p,\theta}(\mathbb{R}^n)$ and the inequalities

$$\|\nabla g\|_\infty \leq \|g\|_{W^1_\infty(\mathbb{R}^n)} \quad \textit{and} \quad \|f_1''\|_{F^{s-2}_{p,q}(\mathbb{R})} \leq c \, \|f'\|_{F^{s-1}_{p,q}(\mathbb{R})},$$

then we conclude by the embedding $B^s_{p,\theta}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n)$ and the fact that

$$\begin{aligned}
\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} &\leq c \, \|f_1 \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} + |f'(0)| \, \|g\|_{F^s_{p,q}(\mathbb{R}^n)} \\
&\leq c \, \|f_1 \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} + \|f'\|_\infty \|g\|_{B^s_{p,\theta}(\mathbb{R}^n)}.
\end{aligned}$$

□

P r o o f o f T h e o r e m 5.1 The assertion follows from Proposition 5.2 and the equality

$$f \circ g := (f\rho_t) \circ g \qquad (\textit{for} \quad t \geq \|g\|_\infty). \qquad \square$$

Theorem 5.1 yields an extension of [5, Lemma 3.4] to Triebel–Lizorkin spaces, then we have the result of Bourdaud [5, Thm. 3.1] by the same nonlinear interpolation argument. Namely

Corollary 5.3 *Suppose* (1.2). *Let* $1 \leq r \leq \infty$ *and let a real number* t *be such that* $t > s > 1 + (1/p)$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function such that* $f(0) = 0$ *and* $f \in F^{t,\ell oc}_{p,r}(\mathbb{R})$. *Then the composition operator* T_f *takes* $B^s_{p,q}(\mathbb{R}^n) \cap \mathcal{H}$ *to* $B^s_{p,q}(\mathbb{R}^n)$.

P r o o f. *Step 1.* Suppose that $f' \in F^{t-1}_{p,r}(\mathbb{R})$. Using Proposition 5.2, then for

$$\theta := \frac{1}{t - [t] + 1 - (1/p)},$$

we have

$$\|f \circ g\|_{F^t_{p,r}(\mathbb{R}^n)} \leq c\,\|f'\|_{F^{t-1}_{p,r}(\mathbb{R})}\,(1 + \|g\|_{\mathcal{H}})^{t-1-(1/p)}\,\|g\|_{B^t_{p,\theta}(\mathbb{R}^n)} \quad (\forall g \in B^t_{p,\theta}(\mathbb{R}^n) \cap \mathcal{H}).$$

On the other hand we have

$$\|f \circ g_1 - f \circ g_2\|_p \leq \|f'\|_\infty\,\|g_1 - g_2\|_p \qquad (\forall g_1, g_2 \in L_p(\mathbb{R}^n)).$$

Then by a nonlinear interpolation theorem of Peetre [14] (see also [17, Prop. 2.5.4/2, p. 88]), we obtain

$$\|f \circ g\|_{B^s_{p,q}(\mathbb{R}^n)} \leq c\,\|f'\|_{F^{t-1}_{p,r}(\mathbb{R})}\,(1 + \|g\|_{\mathcal{H}})^{(s/t)(t-1-(1/p))}\,\|g\|_{B^s_{p,q}(\mathbb{R}^n)} \quad (\forall g \in B^s_{p,q}(\mathbb{R}^n) \cap \mathcal{H}).$$

Step 2. If $f \in F^{t,\ell oc}_{p,r}(\mathbb{R})$, using the equality

$$f \circ g := (f\rho_\tau) \circ g \qquad (\textit{for} \quad \tau \geq \|g\|_\infty),$$

then we proceed as in Step 1 of the proof of Proposition 4.1. □

Acknowledgements Thanks to Gérard Bourdaud for his help and his remarks during the preparation of this article. Also thanks to the referee for his remarks which led to an improvement of the paper.

References

[1] D. Adams and M. Frazier, BMO and smooth truncation in Sobolev spaces, Studia Math. **89**, 241–260 (1988).
[2] D. Adams and M. Frazier, Composition operators on potential spaces, Proc. Amer. Math. Soc. **114**, 155–165 (1992).
[3] G. Bourdaud, La trivialité du calcul fonctionnel dans l'espace $H^{3/2}(\mathbb{R}^4)$, C. R. Acad. Sci. Paris, Série I, **314**, 187–190 (1992).
[4] G. Bourdaud, Fonctions qui opèrent sur les espaces de Besov et de Triebel, Ann. Inst. H. Poincaré, Analyse Non Linéaire, **10**, 413–422 (1993).
[5] G. Bourdaud, Une propriété de composition dans l'espace H^s (II), C. R. Acad. Sci. Paris, Série I, **342**, 243–246 (2006).
[6] G. Bourdaud, M. Lanza de Cristoforis, and W. Sickel, Superposition operators and functions of bounded p-variation. Rev. Mat. Iberoamer. **22**, 455–485 (2006).
[7] G. Bourdaud, M. Lanza de Cristoforis, and W. Sickel, Superposition operators and functions of bounded p-variation II, Nonlinear Anal. Ser. A, **62**, 483–518 (2005).
[8] G. Bourdaud, M. Moussai, and W. Sickel, An optimal symbolic calculus on Besov algebras, Ann. I. H. Poincaré – AN **23**, 949–956 (2006).
[9] G. Bourdaud, M. Moussai and W. Sickel, Towards sharp superposition theorems in Besov and Lizorkin–Triebel spaces, Nonlinear Anal. **68**, 2889–2912 (2008).
[10] H. Brezis and P. Mironescu, Gagliardo-Nirenberg, compositon and products in fractional Sobolev spaces, J. Evol. Equ. **1**, 387–404 (2001).
[11] M. Bruneau, Variation Totale d'une Fonction, Lecture Notes in Mathematics Vol. 413 (Springer, Berlin, 1974).

[12] J. Franke, On the spaces $F^s_{p,q}(\mathbb{R})$ of Triebel–Lizorkin type: Pointwise multipliers and spaces on domains, Math. Nachr. **125**, 29–68 (1986).
[13] V. Maz'ya and T. Shaposhnikova, An elementary proof of the Brezis and Mironescu theorem on the compositon operator in fractional Sobolev spaces, J. Evol. Equ. **2**, 113–125 (2002).
[14] J. Peetre, Interpolation of Lipschitz operators and metric spaces, Mathematica (Cluj) **12**, 325–334 (1970).
[15] J. Peetre, New Thoughts on Besov Spaces (Duke University Mathematics Series I, Durham, NC, 1976).
[16] T. Runst, Mapping properties of non-linear operators in spaces of Triebel–Lizorkin and Besov type, Analysis Mathematica **12**, 313–346 (1986).
[17] T. Runst and W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations (De Gruyter, Berlin, 1996).
[18] W. Sickel, Necessary condition on composition operators acting on Sobolev spaces of fractional order. The critical case $1 < s < (n/p)$, Forum Math. **9**, 267–302 (1997).
[19] H. Triebel, Theory of Function Spaces (Birkhäuser, Basel, 1983).
[20] H. Triebel, Theory of Function Spaces II (Birkhäuser, Basel, 1992).