

# Characterization of Periodic Function Spaces via Means of Abel–Poisson and Bessel–Potential Type

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## 1. INTRODUCTION

Let  $f \in L_1(T_n)$ ,  $T_n$  an  $n$ -dimensional torus, be represented by its Fourier series

$$f = \sum_{k \in Z_n} \hat{f}(k) e^{ikx}. \tag{1}$$

For  $\psi = \psi(\xi)$ ,  $\xi \in R_n$ , an appropriate bounded function defined on the Euclidean  $n$ -space  $R_n$ , with  $\psi(0) = 1$  we consider the approximation of  $f$  by means

$$M_\nu^\psi f(x) = \sum_{k \in Z_n} \psi\left(\frac{k}{\nu}\right) \hat{f}(k) e^{ikx}, \quad \nu = 1, 2, \dots, \tag{2}$$

for  $\nu \rightarrow \infty$ . Most of the classical means are included in this setting. For example, if  $\psi = e^{-|\xi|^\theta}$ ,  $0 < \theta < \infty$ , or if  $\psi = (1 + |\xi|^2)^{-\beta/2}$ ,  $0 < \beta < \infty$ , the means  $M_\nu^\psi f$  coincide with the Abel–Poisson (Abel–Cartwright) and Bessel–potential means, respectively. Generating functions  $\psi$  having compact support lead to partial sums, de la Vallée–Poussin or Riesz means which have been considered in [16]. In this sense the present paper is the continuation of [16]. The rate of convergence of the means  $M_\nu^\psi f$  against  $f$  is measured by

$$\sum_{\nu=1}^{\infty} \nu^{sq-1} \|f - M_\nu^\psi f\|_{L_p(T_n)}^q < \infty \tag{3}$$

and by

$$\left\| \left( \sum_{\nu=1}^{\infty} \nu^{sq-1} |f(x) - M_\nu^\psi f(x)|^q \right)^{1/q} \right\|_{L_p(T_n)} < \infty, \tag{4}$$

where  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s > 0$  ( $(\sum_v v^{-1}(\dots)^q)^{1/q}$  replaced by  $\sup_v (\dots)$  if  $q = \infty$ ). It turns out that (3) and (4) are related to the periodic Besov spaces  $B_{p,q}^s(T_n)$  and to the periodic Lizorkin–Triebel spaces  $F_{p,q}^s(T_n)$  (for definitions see Section 2). For a certain range of parameters  $p$ ,  $q$ , and  $s$  depending on the smoothness of  $\psi$ , and of the decay of the functions  $1 - \psi(\xi)$  for  $|\xi| \rightarrow 0$  and  $\psi(\xi)$  for  $|\xi| \rightarrow \infty$ , we are able to give an equivalent characterization of the classes of all functions satisfying (3) or (4) within the scope of these two scales of spaces. Problem (3) for Abel–Poisson and Bessel-potential means has been investigated by many authors in the cases  $1 \leq p \leq \infty$ ,  $q = \infty$ , and  $s > 0$ . We refer to P. L. Butzer and R. J. Nessel [3, Chaps. 3 and 12], B. I. Golubov [7], W. Trebels [25], and M. Zamansky [31]. Results concerning (3) can also be obtained using the comparison theorems by H. S. Shapiro [18–20], J. Boman and H. S. Shapiro [2], and J. Boman [1]; cf. also W. Trebels [25]. Our main interest here is the extension to values  $p, q$  less than 1 and to spaces of Hardy–Sobolev type. In particular, the characterization of (4) in the case  $\min(p, q) \leq 1$  requires the use of some recent developments in the theory of the spaces  $F_{p,q}^s(T_n)$ . We need some powerful techniques based on inequalities of Nikol’skij type and vector-valued inequalities for the Hardy–Littlewood maximal function. The standard methods that work in the case  $p \geq 1$  and are used to treat problem (3) cannot be applied here. We refer also to the negative results concerning pointwise comparison by W. Dickmeis, R. J. Nessel, and E. van Wickeren [4]. Let us add that (4) is related to the problem of strong summability, cf., for example, L. Leindler [8, 9]. The connection with the spaces  $F_{p,q}^s(T_n)$  was pointed out in [15, 21, 22].

Let us also mention that problem (3) in the framework of Besov spaces (nonperiodic case,  $p \geq 1$ ) is treated in J. Peetre [14, cf. in particular Chap. 8, and Chap. 11 for some remarks concerning  $p < 1$ ].

The paper is organized as follows. In Section 2 we shall give the definitions and properties of the spaces  $B_{p,q}^s(T_n)$  and  $F_{p,q}^s(T_n)$  and deal with the means  $M_v^\psi f$ . The main parts of the paper are contained in Section 3 and Section 4. In Section 3 general results concerning (4) and the spaces  $F_{p,q}^s(T_n)$  are stated. The interplay between (3) and the spaces  $B_{p,q}^s(T_n)$  is considered in Section 4. Finally, these general results shall be applied to derive results for Abel–Poisson, Bessel-potential, and Riesz means in Section 5.

Throughout the paper,  $c, c', \dots$  denote constants which may be different at each appearance, possibly depending on the dimension or other parameters.

2. PRELIMINARIES

2.1 Notations

We adopt the notation from [17, Chap. 3].  $Z_n, R_n, T_n$  have the usual meaning.  $D'(T_n)$  and  $S'(R_n)$  stand for the distributions on  $T_n$  and the tempered distributions on  $R_n$ , respectively.  $D'(T_n)$  is identified with  $S'_n(R_n)$  ( $2\pi$ -periodic distributions on  $R_n$ ). We put

$$(F\varphi)(y) = (2\pi)^{-n/2} \int_{R_n} \varphi(x) e^{-ixy} dx, \quad y \in R_n \tag{5}$$

(Fourier transform of  $\varphi \in S(R_n)$  and

$$\hat{f}(k) = (2\pi)^{-n} f(e^{-ikx}), \quad k \in Z_n \tag{6}$$

(Fourier coefficients of  $f \in D'(T_n)$ ).  $L_p, 0 < p \leq \infty$ , and  $C$  mean the  $p$ -integrable and continuous functions on  $T_n$ , respectively. Their (quasi-) norms are denoted by  $\|\cdot\|_{L_p}$  and  $\|\cdot\|_C$ . By  $L_p(R_n), 0 < p \leq \infty$ , we mean the non-periodic counterparts.

Finally,  $g * f$  stands for the convolution of  $f \in S'(R_n)$  and  $g \in S'(R_n)$  whenever it makes sense. For the general background concerning periodic distributions we refer to [5, Chap. 12].

2.2 The Means  $M_\nu^\psi f, \nu = 1, 2, \dots$

Let  $\psi = \psi(y) \in L_\infty(R_n)$  and let  $\psi$  be defined for all  $y \in R_n$ , where  $\psi(0) = 1$ . For  $f \in D'(T_n)$  we introduce

$$M_\nu^\psi f(x) = \sum_{k \in Z_n} \psi\left(\frac{k}{\nu}\right) \hat{f}(k) e^{ikx}, \quad \nu = 1, 2, \dots \tag{7}$$

$M_\nu^\psi f$  has to be understood in the sense of convergence in  $D'(T_n)$  or in  $S'(R_n)$ . Both make sense and we have

$$M_\nu^\psi f(x) = c[F^{-1}\psi(\nu^{-1}\cdot)] * f = c[\nu^n(F^{-1}\psi)(\nu\cdot)] * f. \tag{8}$$

If  $f \in L_p, 1 \leq p \leq \infty$ , and if  $F^{-1}\psi \in L_1(R_n)$  then according to (8)  $M_\nu^\psi f \in L_p$  and

$$(M_\nu^\psi f)(x) = c \int_{R_n} [F^{-1}\psi(\nu^{-1}\cdot)](y) f(x-y) dy, \quad \nu = 1, 2, \dots \tag{9}$$

If  $\psi$  is continuous in a neighbourhood of 0 then

$$M_\nu^\psi f \rightarrow f \quad (\nu \rightarrow \infty)$$

in  $D'(T_n)$  or  $S'(R_n)$  for all  $f \in D'(T_n)$ . This describes an approximation

process by periodic functions, in general, and by trigonometric polynomials if  $\text{supp } \psi$  is compact.

In order to prove our main results we need the Hardy–Littlewood maximal function which is defined by

$$(Mf)(x) = \sup \frac{1}{|Q|} \int_Q |f(y)| dy$$

for any locally integrable function  $f$ . Here the supremum is taken over all cubes  $Q$  centered at  $x$  with sides parallel to the coordinate axes. Of great use are the following inequalities which we adopt from [16, Lemma 2 and Lemma 3].

LEMMA 1. *Let  $0 < r \leq 1$ . Let  $\rho(\xi)$  be a continuous function with  $F^{-1}\rho \in L_1(R_n)$  and  $\text{supp } \rho \subset \{\xi \mid |\xi| \leq 4\}$ . Let*

$$f(x) = \sum_{|k| < 2^{j+K}} \hat{f}(k) e^{ikx}, \quad j, K = 0, 1, 2, \dots,$$

be a trigonometric polynomial of radial degree  $2^{j+K}$  ( $|k|^2 = (k_1^2 + \dots + k_n^2)$ ).

(i) *There exists a positive constant  $c$  independent of  $f$ ,  $\rho$ ,  $j$ , and  $K$  such that*

$$\left| \sum_{k \in Z_n} \rho(2^{-j}k) \hat{f}(k) e^{ikx} \right|^r \leq c 2^{(K+j)n(1-r)} \int_{R_n} |(F^{-1}\rho(2^{-j}\cdot))(y)|^r |f(x-y)|^r dy \quad (10)$$

holds for all  $x \in T_n$ .

(ii) *Let*

$$|(F^{-1}\rho)(y)| \leq c(1 + |y|)^{-\lambda} \quad (11)$$

for all  $y \in R_n$  and for some real number  $\lambda > n/r$ . Then there exists a constant  $c' = c'(c, \lambda)$  independent of  $f$ ,  $j$ , and  $K$  such that

$$\left| \sum_{k \in Z_n} \rho(2^{-j}k) \hat{f}(k) e^{ikx} \right|^r \leq c' 2^{Kn(1-r)} M|f|^r(x) \quad (12)$$

holds for all  $x \in R_n$ , and all  $\rho(\xi)$  satisfying (11).

Remark 1. If  $r = 1$  then we can replace the trigonometric polynomial  $f$  by an arbitrary locally integrable function in (10) and (12) as far as the right-hand sides make sense almost everywhere.

2.3. The Spaces  $B_{p,q}^s$  and  $F_{p,q}^s$

Let  $\Phi$  be the class of systems of test functions with the following properties: There exist positive constants  $c_0, c_1,$  and  $c_2$  with  $c_1 \leq 1$  such that

$$\begin{aligned} \text{supp } \varphi_0 &\subset \{ \xi \mid |\xi| \leq c_0 \} \\ \text{supp } \varphi_k &\subset \left\{ \xi \mid c_1 2^k \leq |\xi| \leq \frac{1}{c_1} 2^k \right\} \\ 1 &= \sum_{k=0}^{\infty} \varphi_k(\xi), \quad \xi \in R_n \end{aligned}$$

and

$$|D^\alpha \varphi_k(x)| \leq c_2 2^{-k|\alpha|}$$

for each multi-index  $\alpha$  and each  $k, k=0, 1, \dots$

DEFINITION. Let  $0 < p \leq \infty, 0 < q < \infty, -\infty < s < \infty.$  Let  $\{\varphi_j\}_{j=0}^\infty \in \Phi.$  Let  $f \in D'(T_n)$  and

$$f_j(x) = \sum_{k \in Z_n} \varphi_j(k) \hat{f}(k) e^{ikx}; \quad j=0, 1, 2, \dots \tag{13}$$

(i) We put

$$\begin{aligned} B_{p,q}^s &= \left\{ f \mid f \in D'(T_n), \|f\|_{B_{p,q}^s}^\varphi \right. \\ &= \left. \left( \sum_{j=0}^{\infty} 2^{jsq} \|f_j(x)\|_{L_p}^q \right)^{1/q} < \infty \right\}, \end{aligned} \tag{14}$$

$$\begin{aligned} B_{p,\infty}^s &= \{ f \mid f \in D'(T_n), \|f\|_{B_{p,\infty}^s}^\varphi \\ &= \sup_{j=0,1,\dots} 2^{js} \|f_j(x)\|_{L_p} < \infty \}. \end{aligned} \tag{15}$$

(ii) If additionally  $p < \infty,$  then we put

$$\begin{aligned} F_{p,q}^s &= \left\{ f \mid f \in D'(T_n), \|f\|_{F_{p,q}^s}^\varphi \right. \\ &= \left. \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |f_j(x)|^q \right)^{1/q} \right\|_{L_p} < \infty \right\}, \end{aligned} \tag{16}$$

$$\begin{aligned} F_{p,\infty}^s &= \{ f \mid f \in D'(T_n), \|f\|_{F_{p,\infty}^s}^\varphi \\ &= \left\| \sup_{j=0,1,\dots} 2^{js} |f_j(x)| \right\|_{L_p} < \infty \}. \end{aligned} \tag{17}$$

*Remark 2.* For a detailed discussion of these function spaces we refer to [17, Chap. 3]. There further characterizations, e.g., by differences and derivatives can be found (cf. also [26] and [28] for the non-periodic analog). Note that the spaces  $F_{p,q}^m$ ,  $m = 1, 2, \dots, 1 < p < \infty$  coincide with the periodic Sobolev spaces.

### 3. GENERAL RESULTS FOR $F_{p,q}^s$

#### 3.1. Direct Results

Let  $h(\xi) \in S(R_n)$  and  $H(\xi) \in S(R_n)$  be functions satisfying

$$\begin{aligned} h(\xi) &= 1 && \text{if } |\xi| \leq 1, \\ \text{supp } h &\subset \{ \xi \mid |\xi| \leq 2 \} \end{aligned} \tag{18}$$

and

$$\begin{aligned} H(\xi) &= 1 && \text{if } \frac{1}{2} \leq |\xi| \leq 2 \\ \text{supp } H &\subset \left\{ \xi \mid \frac{1}{4} \leq |\xi| \leq 4 \right\}. \end{aligned} \tag{19}$$

**THEOREM 1.** *Let  $\psi$  be a bounded continuous function on  $R_n$  with  $\psi(0) = 1$  and  $F^{-1}\psi \in L_1(R_n)$ . Let  $\sigma > 0$  and  $\lambda > n$  such that*

$$\sup_{l=0, -1, -2, \dots} |F^{-1}[|2^l \xi|^{-\sigma} (1 - \psi(2^l \xi)) H(\xi)](y)| \leq c(1 + |y|)^{-\lambda} \tag{20}$$

and

$$\sup_{l=1, 2, \dots} |F^{-1}[\psi(2^l \xi) H(\xi)](y)| \leq c(1 + |y|)^{-\lambda} \tag{21}$$

hold for all  $y \in R_n$ .

If  $n/\lambda < p < \infty$ ,  $n/\lambda < q < \infty$ , and  $0 < s < \sigma$  then there exists a positive constant  $c'$  such that

$$\left\| \left( \sum_{v=1}^{\infty} v^{sq-1} |f(x) - M_v^\psi f(x)|^q \right)^{1/q} \Big| L_p \right\| \leq c \|f\|_{F_{p,q}^s} \tag{22}$$

holds for all  $f \in F_{p,q}^s \cap L_1$  (modification if  $q = \infty$  by  $\sup_v v^s |\dots|$ ).

*Proof.* Step 1. Let  $f \in F_{p,q}^s \cap L_1$ ,  $s > 0$ . Then  $f \in L_p$ . Furthermore  $M_v^\psi f \in L_p$ , cf. (9). We decompose

$$\begin{aligned} \psi(\xi) &= \psi(\xi) h(2^{-L}\xi) + \psi(\xi)(1 - h(2^{-L}\xi)) \\ &= \eta(\xi) + \chi(\xi), \end{aligned} \tag{23}$$

where  $L$  is a natural number to be chosen later on. The idea is to “approximate” the means  $M_\nu^\psi f$  by the “truncated” means  $M_\nu^\chi f$ . We have

$$\begin{aligned} & \left\| \left( \sum_{\nu=1}^{\infty} \nu^{sq-1} |f(x) - M_\nu^\psi f(x)|^q \right)^{1/q} \right\|_{L_p} \\ & \leq c \left( \left\| \left( \sum_{\nu=1}^{\infty} \nu^{sq-1} |f(x) - M_\nu^\eta f(x)|^q \right)^{1/q} \right\|_{L_p} \right. \\ & \quad \left. + \left\| \left( \sum_{\nu=1}^{\infty} \nu^{sq-1} |M_\nu^\chi f(x)|^q \right)^{1/q} \right\|_{L_p} \right), \end{aligned} \tag{24}$$

where  $M_\nu^\chi f$  has the meaning of (7) with  $\chi$  instead of  $\psi$ . We estimate the first summand on the right-hand side of (24). Using formulas (4.6) and (4.7) from [16] we obtain

$$\begin{aligned} & \sum_{\nu=1}^{\infty} \nu^{sq-1} |f(x) - M_\nu^\eta f(x)|^q \\ & \leq c \sum_{j=0}^{\infty} 2^{jsq} \sup_{1 < \tau \leq 2} \left| \sum_{k \in \mathbb{Z}_n} \eta_\tau(2^{-j}k) \hat{f}(k) e^{ikx} \right|^q, \end{aligned} \tag{25}$$

where

$$\eta_\tau(\xi) = \eta(\xi/2) - \eta(\tau\xi/2).$$

Now, let us choose a function  $\varphi \in S(\mathbb{R}_n)$  such that

$$\begin{aligned} & \text{supp } \varphi \subset \left\{ \xi \mid \frac{1}{4} \leq |\xi| \leq 1 \right\}, \\ & \sum_{l=1}^{\infty} \varphi(2^{-l}\xi) = 1 \quad \text{if } |\xi| \geq 1. \end{aligned}$$

It is not hard to see that the system  $\{\varphi_k\}_{k=0}^{\infty}$ , defined by

$$\varphi_k(\cdot) = \varphi(2^{-k}\cdot), \quad k = 1, 2, \dots,$$

and an appropriate function  $\varphi_0$ , is an element of  $\Phi$ . By the properties of  $\varphi$ ,  $h$ , and  $H$  we have

$$\begin{aligned} \eta_\tau(2^{-j}k) &= \sum_{l=1}^{j+L+2} \eta_\tau(2^{-j}k) \varphi(2^{-l}k) H(2^{-l}k) \\ &= \sum_{l=-\infty}^{L+2} \eta_\tau(2^{-j}k) \varphi_{l+j}(k) H_{l+j}(k). \end{aligned} \tag{26}$$

Here  $H_l(\cdot) = H(2^{-l}\cdot)$  and  $\varphi_{-l} = H_{-l} = 0$  if  $l = 0, 1, 2, \dots$ . We put

$$f_{l,\sigma}(x) = \sum_{k \in \mathbb{Z}_n} |2^{-l}k|^\sigma \varphi_l(k) \hat{f}(k) e^{ikx}, \quad l = 0, \pm 1, \dots$$

By (30) we obtain

$$\begin{aligned} & \left| \sum_{k \in \mathbb{Z}_n} \eta_\tau(2^{-j}k) \hat{f}(k) e^{ikx} \right| \\ &= \sum_{l=-\infty}^{L+2} 2^{l\sigma} \left| \sum_{k \in \mathbb{Z}_n} |2^{-j}k|^{-\sigma} H_{j+l}(k) \hat{f}_{j+l,\sigma}(k) e^{ikx} \right|. \end{aligned} \tag{27}$$

The function  $\psi$  satisfies (20) and (21). Therefore, inequality (11) holds for all functions  $\rho(\xi) = |2^l\xi|^{-\sigma} \eta_\tau(2^l\xi) H(\xi)$ ,  $1 < \tau \leq 2$ , where the constant  $c$  is independent of  $L, l$ , and  $\tau$ ,  $l \leq L+2$ ,  $1 < \tau \leq 2$ . This is not difficult to see; we omit the details. Hence we can apply part (ii) of Lemma 1 to the right-hand side of (27). This gives

$$\begin{aligned} & \left| \sum_{k \in \mathbb{Z}_n} \eta_\tau(2^{-j}k) \hat{f}(k) e^{ikx} \right| \\ & \leq c \sum_{l=-\infty}^{L+2} 2^{l\sigma} (M |f_{j+l,\sigma}|^r)^{1/r}(x), \quad \frac{n}{\lambda} < r < 1, \end{aligned} \tag{28}$$

where the constant  $c$  is independent of  $j, L, l, \tau, x$ , and  $f$ . (25) and (28) lead to

$$\begin{aligned} & \sum_{v=1}^{\infty} v^{sq-1} |f(x) - M_v^n f(x)|^q \\ & \leq c \sum_{j=0}^{\infty} 2^{jsq} \left( \sum_{l=-\infty}^{L+2} 2^{l\sigma} (M |f_{j+l,\sigma}|^r(x))^{q/r} \right). \end{aligned} \tag{29}$$

This is the counterpart of formula (4.12) in [16]. By our assumptions concerning  $\lambda, p$ , and  $q$  we may choose  $r < \min(1, p, q)$ . Then

$$\begin{aligned} & \left\| \left( \sum_{v=1}^{\infty} v^{sq-1} |f(x) - M_v^n f(x)|^q \right)^{1/q} \right\|_{L^p}^r \\ & \leq c \left( \sum_{l=-\infty}^{L+2} 2^{l(\sigma-s)r} \left\| \left( \sum_{j=0}^{\infty} (M |2^{(j+l)s} f_{j+l,\sigma}|^r(x))^{q/r} \right)^{r/q} \right\|_{L_{p/r}} \right) \\ & \leq c' \left\| \left( \sum_{j=0}^{\infty} (M |2^{js} f_{j,\sigma}|^r(x))^{q/r} \right)^{r/q} \right\|_{L_{p/r}}, \end{aligned} \tag{30}$$



where the last inequality follows by the assumption  $s < \sigma$ . Applying the periodic version of the vector-valued maximal inequality of C. Fefferman and E. M. Stein [6] (cf. [17, Proposition 3.2.4]) to the right-hand side of (30) we obtain

$$\begin{aligned} & \left\| \left( \sum_{v=1}^{\infty} v^{sq-1} |f(x) - M_v^n f(x)|^q \right)^{1/q} \middle| L_p \right\| \\ & \leq c \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |f_{j,\sigma}(x)|^q \right)^{1/q} \middle| L_p \right\|. \end{aligned} \tag{31}$$

Repeating the above arguments with  $|\xi|^\sigma \varphi(\xi)$  instead of  $|\xi|^{-\sigma} \eta_\tau(\xi)$  and using that

$$\varphi_j(x) = [\varphi_{j-1}(x) + \varphi_j(x) + \varphi_{j+1}(x)] \varphi_j(x),$$

the right-hand side of (31) can be estimated by  $c \|f\|_{F_{p,q}^s}$ .

*Step 2.* We estimate the second summand on the right-hand side of (24). Obviously

$$\begin{aligned} & \sum_{v=1}^{\infty} v^{sq-1} |M_v^\chi f(x)|^q \\ & \leq c \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} |M_v^\chi f(x)|^q \\ & \leq c \sum_{j=0}^{\infty} 2^{jsq} \sup_{1/2 \leq \tau < 1} \left| \sum_{k \in \mathbb{Z}_n} \chi(2^{-j}\tau k) \hat{f}(k) e^{ikx} \right|^q. \end{aligned} \tag{32}$$

Let  $\{\varphi_j\}_{j=0}^\infty$  be the system of Step 1. Then (19) and (20) yield

$$\chi(2^{-j}\tau\xi) = \sum_{l=L-2}^{\infty} \chi(2^l\tau\xi) H_{l+j}(\xi) \varphi_{l+j}(\xi), \quad j=0, 1, \dots$$

We put

$$\rho_{\tau,l}(\xi) = \chi(2^l\tau\xi) H(\xi), \quad l=L-2, L-1, \dots, \quad \frac{1}{2} \leq \tau \leq 1.$$

The function  $\psi$  satisfies (21). Consequently, using the properties of  $h$  and  $H$ , we can verify that the above functions  $\rho_{\tau,l}(\xi)$  satisfy condition (11), where the constant  $c$  does not depend on  $\tau$ ,  $\frac{1}{2} \leq \tau \leq 1$ , and  $l=L-2$ ,

$L - 1, \dots$ . Therefore we can apply Lemma 1, part (ii), to estimate the right-hand side of (32). We arrive at

$$\begin{aligned} & \left\| \left( \sum_{\nu=1}^{\infty} \nu^{sq-1} |M_{\nu}^{\lambda} f(x)|^q \right)^{1/q} \Big| L_p \right\|^r \\ & \leq c \sum_{l=L-2}^{\infty} 2^{-lrs} \left\| \left( \sum_{j=0}^{\infty} (M |2^{(j+l)s} f_{j+l}|^r)^{q/r} \right)^{r/q} \Big| L_{p/r} \right\|^r, \end{aligned} \tag{33}$$

where  $r$  can be chosen such that  $n/\lambda < r < \min(1, p, q)$ . The constant  $c$  is independent of  $r$  and  $L$ . Using the assumption  $s > 0$  we obtain from (33) in the same way as in Step 1 (cf. (30) and (31)) that

$$\left\| \left( \sum_{\nu=1}^{\infty} \nu^{sq-1} |M_{\nu}^{\lambda} f(x)|^q \right)^{1/q} \Big| L_p \right\| \leq c 2^{-Ls} \|f\|_{F_{p,q}^s} \tag{34}$$

for all admitted parameters  $p, q, s$ . Using (24) and the estimate from Step 1 this completes the proof of (22).

*Remark 3.* Let us discuss the conditions (20) and (21). To this end we introduce the non-periodic Nikol'skij (Besov–Lipschitz) space  $B_{1,\infty}^{\lambda}(R_n)$ . Let  $\lambda = l + r, l = 0, 1, 2, \dots, 0 < r \leq 1$ . Then

$$\begin{aligned} B_{1,\infty}^{\lambda}(R_n) &= \left\{ f \mid f \in L_1(R_n), \|f\|_{B_{1,\infty}^{\lambda}(R_n)} \right. \\ &= \|f\|_{L_1(R_n)} + \sum_{|\alpha|=l} \sup_{|h| \neq 0} |h|^{-r} \\ &\quad \left. \times \|\Delta_h^{[\alpha]+1} D^{\alpha} f\|_{L_1(R_n)} < \infty \right\}. \end{aligned} \tag{35}$$

The proof of [16, Corollary 2] shows that (20) and (21) are satisfied if

$$\sup_{l=0, -1, -2, \dots} \| |2^l x|^{-\sigma} (1 - \psi(2^l x)) H(x) \|_{B_{1,\infty}^{\lambda+\varepsilon}(R_n)} < \infty \tag{36}$$

and

$$\sup_{l=1, 2, \dots} \| \psi(2^l x) H(x) \|_{B_{1,\infty}^{\lambda+\varepsilon}(R_n)} < \infty \tag{37}$$

for some number  $\varepsilon, \varepsilon > 0$ . Moreover, if

$$|(F^{-1}\psi)(y)| \leq c(1 + |y|)^{-\lambda} \tag{38}$$

then

$$\sup_{0 < |x| \leq \delta_1} \sum_{|\alpha|=m} |x|^{|\alpha|} |D^{\alpha}(|x|^{-\sigma} (1 - \psi(x)))| < \infty \tag{39}$$

and

$$\sup_{|x| \geq \delta_2} \sum_{|\alpha|=m} |x|^{|\alpha|} |D^\alpha \psi(x)| < \infty, \tag{40}$$

where  $m$  is a natural number larger than  $\lambda$ , imply (20) and (21), respectively. Here  $0 < \delta_1 < \infty$ ,  $0 < \delta_2 < \infty$ . Note that (39) necessarily leads to

$$1 - \psi(x) = 0(|x|^\sigma) \quad (|x| \rightarrow 0). \tag{41}$$

It will be seen later on that  $\sigma$  corresponds to the saturation order of the approximation process  $M_v^\psi f$ . Furthermore, the number  $\lambda$  is related to the smoothness properties of the function  $\psi$  involved. Note that we do not need any smoothness properties of  $\psi$  at the point 0 itself but on  $R_n \setminus \{0\}$ . This is an improvement of [16, Theorem 3]. For consequences we refer to Section 5.3 (Riesz means).

*Remark 4.* Let us mention that

$$F_{p,q}^s \cap L_1 = F_{p,q}^s$$

if  $s > n(1/\min(1, p) - 1)$ , cf. [17, Theorems 3.5.1/1].

*Remark 5.* If we restrict ourselves to the case  $1 < p < \infty$  and  $1 < q < \infty$  we can apply vector-valued multiplier theorems. This simplifies the above proof in this case. We refer to the method used in [15] and in [17, Chapter 3], as well as to the results by H. Dappa and W. Trebels [30].

### 3.2. Inverse Results

**THEOREM 2.** *Let  $\psi(\xi)$  be continuous at 0 and infinitely differentiable on  $R_n \setminus \{0\}$  with  $\psi(0) = 1$  and  $F^{-1}\psi \in L_1(R_n)$ . Let  $d$  be a natural number such that*

$$|\psi(\xi/d) - \psi(\xi)| \neq 0 \quad \text{if} \quad \frac{1}{2} \leq |\xi| \leq 4 \tag{42}$$

and let (21) be satisfied for a real number  $\lambda$ ,  $\lambda > n$ . If  $n/\lambda < p < \infty$ ,  $n/\lambda < q \leq \infty$ , and  $n(1/\min(1, p, q) - 1) < s < \infty$  then there exists a positive constant  $c$  such that

$$\|f\|_{F_{p,q}^s} \leq c \left( \|f\|_{L_p} + \left\| \left( \sum_{v=1}^{\infty} v^{sq-1} |f(x) - M_v^\psi f(x)|^q \right)^{1/q} \right\|_{L_p} \right) \tag{43}$$

holds for all  $f \in F_{p,q}^s$  (modification if  $q = \infty$ ).

*Proof.* Without loss of generality we can assume that  $\hat{f}(0)=0$  (cf. Step 1 of the proof of [16, Theorem 1]). In contrast to the proof of Theorem 1 we use another system  $\{\varphi_j\}_j$  to define an appropriate quasinorm in  $F_{p,q}^s$ . Let  $\omega, \varphi \in S(\mathbb{R}_n)$  be functions with the properties

$$\text{supp } \varphi \subset \left\{ \xi \mid \frac{1}{2} \leq |\xi| \leq 2 \right\},$$

$$\text{supp } \omega \subset \{ \xi \mid |\xi| \leq 1 \},$$

and

$$\omega(x) + \sum_{j=1}^{\infty} \varphi(2^{-j+1}x) = 1.$$

Then we put  $\varphi_0(x) = \omega(x)$  and  $\varphi_k(x) = \varphi(2^{-k+1}x)$ ,  $k=1, 2, \dots$ . Hence  $\{\varphi_j\}_{j=0}^{\infty} \in \Phi$  and  $\hat{f}(0)=0$  leads to the inequality

$$\|f\|_{F_{p,q}^s} \leq c \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |f_{j+1}(x)|^q \right)^{1/q} \right\|_{L_p}, \quad (44)$$

where  $c$  is independent of  $f$ .

Let us put  $\eta(\xi) = \psi(\xi/d) - \psi(\xi)$ . Because of the smoothness properties of  $\psi$  and (42) the functions

$$\rho_{\tau}(\xi) = \frac{\varphi(\xi)}{\eta(\tau\xi)}, \quad 1 \leq \tau < 2,$$

are infinitely differentiable and have compact support. It holds that

$$f_{j+1}(x) = \sum_{k \in \mathbb{Z}_n} \rho_{\tau}(2^{-j}k) \eta(\tau 2^{-j}k) \hat{f}(k) e^{ikx}, \quad j=0, 1, \dots \quad (45)$$

Let  $h$  be the function from (18) and let

$$g_{\tau,j}(x) = \sum_{k \in \mathbb{Z}_n} \eta(\tau 2^{-j}k) h(\tau 2^{-L-j}k) \hat{f}(k) e^{ikx}, \quad (46)$$

where  $L$  is a large natural number which will be chosen later on. As a consequence of (45), (46), and part (ii) of Lemma 1 we obtain the estimate

$$\begin{aligned} |f_{j+1}(x)|^r &= \left| \sum_{k \in \mathbb{Z}_n} \rho_{\tau}(2^{-j}k) \hat{g}_{\tau,j}(k) e^{ikx} \right|^r \\ &\leq c 2^{Ln(1-r)} (M |g_{\tau,j}|^r)(x), \end{aligned} \quad (47)$$

where  $n/\lambda < r \leq 1$  and  $c$  is independent of  $t, j$ , and  $L$ . If  $\tau = 2^j/v, v = 2^{j-1}, \dots, 2^j - 1$ , then we put ( $v = 1$  if  $j = 0$ )

$$g_v(x) = g_{\tau,j}(x) = \sum_{k \in Z_n} \eta(v^{-1}k) h(v^{-1}2^{-L}k) \hat{f}(k) e^{ikx}.$$

Now, (47) leads to

$$|f_{j+1}(x)|^q \leq c 2^{Ln(1/r-1)q} 2^{-j} \sum_{v=2^{j-1}}^{2^j-1} (M |g_v|^r)^{q/r}(x)$$

and hence to

$$\begin{aligned} & \left( \sum_{j=0}^{\infty} 2^{jsq} |f_{j+1}(x)|^q \right)^{1/q} \\ & \leq c 2^{Ln(1/r-1)} \left( \sum_{v=1}^{\infty} (M |v^{s-1/q} g_v(x)|^r)^{q/r} \right)^{1/q}. \end{aligned} \tag{48}$$

We can choose  $r < \min(1, p, q)$ . In the same way as in the proof of Theorem 1 (cf. (30), (31)) we derive from (48)

$$\begin{aligned} & \left\| \left( \sum_{j=1}^{\infty} 2^{jsq} |f_j(x)|^q \right)^{1/q} \right\|_{L_p} \\ & \leq c 2^{Ln(1/r-1)} \left\| \left( \sum_{v=1}^{\infty} v^{sq-1} |g_v(x)|^q \right)^{1/q} \right\|_{L_p}. \end{aligned} \tag{49}$$

We split

$$\begin{aligned} |g_v(x)| &= \left| - \sum_{k \in Z_n} \eta(v^{-1}k) \hat{f}(k) e^{ikx} \right. \\ & \quad \left. + \sum_{k \in Z_n} \eta(v^{-1}k) (1 - h(v^{-1}2^{-L}k)) \hat{f}(k) e^{ikx} \right| \\ & \leq |f(x) - M_v^\psi f(x)| + |f(x) - M_{d,v}^\psi f(x)| + |M_v^\chi f(x)|, \end{aligned} \tag{50}$$

where  $\chi(\xi) = (\psi(\xi) - \psi(\xi/d))(1 - h(2^{-L}\xi))$ . It is not difficult to see that (21) is satisfied with  $\chi$  instead of  $\psi$  independent of  $L$  (cf. (32), (33)). Therefore, Step 2 of the preceding proof shows that (34) holds true for this

function  $\chi$ . Note that the restriction  $s < \sigma$  has not been used there. As a consequence of (34) (with the just defined  $\chi$ ), (49), and (50) we find

$$\begin{aligned} & \left\| \left( \sum_{j=1}^{\infty} 2^{jsq} |f_j(x)|^q \right)^{1/q} \Big| L_p \right\| \\ & \leq c 2^{Ln(1/r-1)-s} \|f\|_{F_{p,q}^s} \\ & \quad + c' \left\| \left( \sum_{v=1}^{\infty} v^{sq-1} |f(x) - M_v^\psi f(x)|^q \right)^{1/q} \Big| L_p \right\|, \end{aligned} \tag{51}$$

where  $c'$  depends on  $L$  and  $r$ . We can choose  $r$  such that  $n(1/r-1) < s$  and hence  $L$  such that  $c 2^{Ln(1/r-1)-s} \leq \varepsilon$  for a given  $\varepsilon > 0$ . This proves that

$$\begin{aligned} & c \left\| \left( \sum_{j=1}^{\infty} 2^{jsq} |f_j(x)|^q \right)^{1/q} \Big| L_p \right\| \\ & \leq \varepsilon \|f\|_{F_{p,q}^s} + c(\varepsilon) \left\| \left( \sum_{v=1}^{\infty} v^{sq-1} |f(x) - M_v^\psi f(x)|^q \right)^{1/q} \Big| L_p \right\|. \end{aligned} \tag{52}$$

Now, (52) with  $\varepsilon < 1$  leads to (43).

#### 4. GENERAL RESULTS FOR $B_{p,q}^s$

##### 4.1. Direct Results

The functions  $H(\xi)$  and  $h(\xi)$  have the same meanings as in (18) and (19).

**THEOREM 3.** *Let  $\psi(\xi)$  be a function satisfying the assumptions of Theorem 1 with real numbers  $\sigma > 0$  and  $\lambda > n$ . If  $n/\lambda < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $0 < s < \sigma$  then there exists a positive constant  $c$  such that*

$$\left( \sum_{v=1}^{\infty} v^{sq-1} \|f(x) - M_v^\psi f(x)\|_{L_p}^q \right)^{1/q} \leq c \|f\|_{B_{p,q}^s} \tag{53}$$

holds for all functions  $f \in B_{p,q}^s \cap L_1$  (modification if  $q = \infty$ ). Under the same assumptions there exists a constant  $c'$  such that

$$\left( \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|f(x) - M_v^\psi f(x)\|_{L_p}^q \right)^{1/q} \leq c' \|f\|_{B_{p,q}^s}. \tag{54}$$

*Proof.* Obviously, (53) is an immediate consequence of (54). To prove (54) we split as in (23) and (24). Then we obtain

$$\begin{aligned} & \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|f(x) - M_v^\psi f(x)\|_{L_p}^q \\ & \leq c \left( \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|f(x) - M_v^\eta f(x)\|_{L_p}^q \right. \\ & \quad \left. + \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|M_v^\chi f(x)\|_{L_p}^q \right). \end{aligned} \tag{55}$$

In the same way as in the proof of Theorem 1, Step 2, the second summand on the right-hand side of (55) can be estimated (with obvious modifications and using the scalar case of the Fefferman–Stein inequality). This gives that

$$\left( \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|M_v^\chi f(x)\|_{L_p}^q \right)^{1/q} \leq c 2^{-Ls} \|f\|_{B_{p,q}^s}. \tag{56}$$

In order to estimate the first summand on the right-hand side of (55) we observe that

$$\begin{aligned} & \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|f(x) - M_v^\eta f(x)\|_{L_p}^q \\ & \leq c \left( \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|M_{2^{j+1}}^\eta f(x) - M_v^\eta f(x)\|_{L_p}^q \right. \\ & \quad \left. + \sum_{j=0}^{\infty} 2^{jsq} \|f(x) - M_{2^{j+1}}^\eta f(x)\|_{L_p}^q \right). \end{aligned} \tag{57}$$

Because  $s > 0$  the second summand on the right-hand side of (57) can be estimated by

$$c \sum_{j=0}^{\infty} 2^{jsq} \|M_{2^{j+1}}^\eta f(x) - M_{2^j}^\eta f(x)\|_{L_p}^q.$$

This follows by a modification of the arguments of Step 1 of the proof in [16, Theorem 3]. Hence, we have by (57)

$$\begin{aligned} & \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|f(x) - M_v^\eta f(x)\|_{L_p}^q \\ & \leq c \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|M_{2^{j+1}}^\eta f(x) - M_v^\eta f(x)\|_{L_p}^q. \end{aligned} \tag{58}$$

This estimate is the counterpart of [16, (4.6)]. Modifying the method of Step 1 of the proof of Theorem 1 (cf., in particular, (29)–(35)) the desired inequality (54) follows from (55)–(58).

4.2. Inverse Results

**THEOREM 4.** *Let  $\psi(\xi)$  be a function satisfying the assumptions of Theorem 2 with real numbers  $\sigma > 0$  and  $\lambda > n$ .*

(i) *If  $n/\lambda < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $n(1/\min(1, p) - 1) < s < \infty$  then there exists a positive constant  $c$  such that*

$$\|f\|_{B_{p,q}^s} \leq c \left( \|f\|_{L_p} + \left( \sum_{v=1}^{\infty} v^{sq-1} \|f(x) - M_v^\psi f(x)\|_{L_p}^q \right)^{1/q} \right) \tag{59}$$

*holds for all  $f \in B_{p,q}^s$  (modification if  $q = \infty$ ).*

(ii) *If  $n/\lambda < p \leq \infty$ ,  $0 < u < \infty$ , and  $n(1/\min(1, p, u) - 1) < s < \infty$  then there exists a positive constant  $c$  such that*

$$\begin{aligned} &\|f\|_{B_{p,q}^s} \\ &\leq c \left( \|f\|_{L_p} + \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |f(x) - M_v^\psi f(x)|^u \right)^{1/u} \Big|_{L_p} \right\|^q \right)^{1/q} \right) \end{aligned} \tag{60}$$

*holds for all  $f \in B_{p,q}^s$  (modification if  $q = \infty$ ).*

*Proof. Step 1.* The proof of (i) follows exactly the line of the proof of Theorem 2. We have to use the scalar Hardy–Littlewood maximal inequality instead of the vector-valued one.

*Step 2.* We prove (60). The functions  $\eta(\xi)$ ,  $\rho_\tau(\xi)$  ( $1 \leq \tau < 2$ ),  $g_{\tau,j}(\xi)$  ( $1 \leq \tau < 2, j = 0, 1, \dots$ ), and  $g_\nu(\xi)$  ( $\nu = 1, 2, \dots$ ) have the same meanings as in the proof of Theorem 2. Lemma 1, part (i), shows that

$$\begin{aligned} |f_{j+1}(x)|^r &= \left| \sum_{k \in \mathbb{Z}_n} \rho_\tau(2^{-j}k) \hat{g}_{\tau,j}(k) e^{ikx} \right|^r \\ &\leq c 2^{(L+j)n(1-r)} \int_{R_n} |(F^{-1}\rho_\tau(2^{-j}\cdot))(y)|^r \\ &\quad \times |g_{\tau,j}(x-y)|^r dy \end{aligned} \tag{61}$$

for all  $x \in R_n$ , where  $r, 0 < r \leq 1$ , is a number to be specified. This is the counterpart to (47) and corresponds to [16, (3.31)]. We choose  $r$ , such



that  $r < \min(1, p, u)$ . There exists a real number  $\lambda > n/r$  for each  $r, 0 < r \leq 1$ , such that

$$|(F^{-1}\rho_\tau(2^{-j}\cdot))(y)| \leq c2^{jn}(1 + |2^jy|)^{-\lambda}, \tag{62}$$

where  $c$  is independent of  $\tau, 1 \leq \tau < 2$ . Splitting the integral in (61) and using (62) we find

$$|f_{j+1}(x)| \leq c2^{Ln(1/r-1)} 2^{jn/r} \times \left( \sum_{l=0}^{\infty} 2^{-\lambda lr} \int_{|y| \leq 2^{-j+l}} |g_{\tau,j}(x-y)|^r dy \right)^{1/r}. \tag{63}$$

We put  $\tau = 2^{j+1}/v$ , where  $v = 2^j, 2^j + 1, \dots, 2^{j+1} - 1$ . Taking the  $u$ th power in (63) and summing up over  $v = 2^{j-1}, \dots, 2^j - 1$  yields

$$|f_{j+1}(x)|^u \leq c2^{(Ln(1/r-1) + jn/r)u} 2^{-j} \times \sum_{v=2^j}^{2^{j+1}-1} \left( \sum_{l=0}^{\infty} 2^{-\lambda lr} \int_{|y| \leq 2^{-j+l}} |g_v(x-y)|^r dy \right)^{u/r}. \tag{64}$$

By means of the triangular inequality it follows therefrom (use  $r < u$  and  $\lambda r - n > 0$ ) that

$$\begin{aligned} |f_{j+1}(x)|^r &\leq c2^{Ln(1/r-1)r + jn - jr/u} \\ &\times \left( \sum_{v=2^j}^{2^{j+1}-1} \left( \sum_{l=0}^{\infty} 2^{-\lambda lr} \int_{|y| \leq 2^{-j+l}} |g_v(x-y)|^r dy \right)^{u/r} \right)^{r/u} \\ &\leq c2^{-Ln(r-1) + jn - jr/u} \\ &\times \sum_{l=0}^{\infty} 2^{-\lambda rl} \int_{|y| \leq 2^{-j+l}} \left( \sum_{v=2^j}^{2^{j+1}-1} |g_v(x-y)|^u \right)^{r/u} dy \\ &\leq c'2^{-Ln(r-1)} \\ &\times \sum_{l=0}^{\infty} 2^{-(\lambda r - n)l} M \left| \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |g_v(\cdot)|^u \right)^{1/u} \right|^r (x) \\ &\leq c''2^{-Ln(r-1)} \\ &\times M \left| \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |g_v|^u \right)^{1/u} \right|^r (x). \end{aligned} \tag{65}$$

Now we can apply the Hardy-Littlewood inequality on  $T_n$  (cf. [17, Proposition 3.2.4]) to obtain that

$$\|f_{j+1}(x)|L_p\| \leq c2^{Ln(1/r-1)} \left\| \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |g_v|^u \right)^{1/u} \right\| L_p \tag{66}$$

and hence

$$\sum_{j=1}^{\infty} 2^{jsq} \|f_j(x)|L_p\|^q \leq c 2^{Ln(1/r-1)} \sum_{j=0}^{\infty} 2^{jsq} \times \left\| \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |g_v(x)|^u \right)^{1/u} \right\|_{L_p}. \quad (67)$$

This is the counterpart of (49). Now we argue in the same way as at the end of the proof of Theorem 2 to derive the inequality (60) from (67). This completes the proof.

*Remark 6.* If  $p > 1$ ,  $0 < q \leq \infty$  then (60) holds true for all  $s > 0$  and for all functions with finite right-hand side. In particular, this leads to the following corollary.

**COROLLARY.** Let  $\psi(\xi)$  be a function such that the assumptions of Theorem 2 are satisfied. Let  $f \in L_{\infty}$ .

(i) If  $0 < u < \infty$ ,  $0 < q \leq \infty$ ,  $0 < s$  then there exists a positive number  $c$  such that

$$\|f|B_{\infty,q}^s\| \leq c \left( \|f|L_{\infty}\| + \left( \sum_{j=0}^{\infty} 2^{jsq} \times \left\| \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |f(x) - M_v^{\psi} f(x)|^u \right)^{1/u} \right\|_{L_{\infty}} \right)^q \right)^{1/q} \quad (68)$$

holds.

(ii) If  $0 < q < \infty$ ,  $s > 0$  then there exist positive numbers  $c$  and  $c'$  such that

$$\|f|B_{\infty,\infty}^s\| \leq c \left( \|f|L_{\infty}\| + \sup_{j=0,1,\dots} 2^{js} \times \left\| \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |f(x) - M_v^{\psi} f(x)|^q \right) \right\|_{L_{\infty}}^{1/q} \right) \leq c' \left( \|f|L_{\infty}\| + \left\| \sum_{v=1}^{\infty} v^{sq-1} |f(x) - M_v^{\psi} f(x)|^q \right\|_{L_{\infty}}^{1/q} \right) \quad (69)$$

holds.

*Remark 7.* (68) and (69) show that the results obtained by L. Leindler [8, 9], L. Leindler and A. Meir [10], J. Nemeth [12], and others concerning the analogous problems for partial sums of one-dimensional Fourier

series, de la Vallée–Poussin means, and Féjer means can be generalized to means  $M_v^\psi f$ . We refer also to [15, 16, 21, and 22].

*Remark 8.* If  $u \leq p$  then we have

$$\begin{aligned} & \left\| \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |f(x) - M_v^\psi f(x)|^u \right)^{1/u} \Big| L_p \right\| \\ & \leq \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} \|f(x) - M_v^\psi f(x)\|_{L_p}^u \right)^{1/u} \\ & \leq \sup_{v=2^j, \dots, 2^{j+1}-1} \|f(x) - M_v^\psi f(x)\|_{L_p}. \end{aligned} \tag{70}$$

Hence the right-hand side of (60) can be estimated by the left-hand side of (54) provided that  $u \leq p$ .

### 5. EXAMPLES

Now we are going to apply the general results of Sections 3 and 4 to concrete (classical) means. For abbreviation let us put

$$\|f\|_{B_{p,q}^s; M^\psi} := \|f\|_{L_p} + \left( \sum_{v=1}^{\infty} v^{sq-1} \|f(x) - M_v^\psi f(x)\|_{L_p}^q \right)^{1/q}, \tag{71}$$

$$\begin{aligned} & \|f\|_{B_{p,q}^s; M^\psi, u} \\ & := \|f\|_{L_p} + \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| \left( 2^{-j} \sum_{v=2^j}^{2^{j+1}-1} |f(x) - M_v^\psi f(x)|^u \right)^{1/u} \Big| L_p \right\|^q \right)^{1/q}, \end{aligned} \tag{72}$$

$$\begin{aligned} & \|f\|_{B_{p,q}^s; M^\psi, * } \\ & := \|f\|_{L_p} + \left( \sum_{j=0}^{\infty} 2^{jsq} \sup_{v=2^j, \dots, 2^{j+1}-1} \|f(x) - M_v^\psi f(x)\|_{L_p}^q \right)^{1/q}, \end{aligned} \tag{73}$$

$$\|f\|_{F_{p,q}^s; M^\psi} := \|f\|_{L_p} + \left\| \left( \sum_{v=1}^{\infty} v^{sq-1} |f(x) - M_v^\psi f(x)|^q \right)^{1/q} \Big| L_p \right\| \tag{74}$$

(modification if  $q = \infty$ ). Given a generating function  $\psi$  we are interested in the admissible range of parameters  $p, q, u$ , and  $s$  such that the above quasi-norms are equivalent to the quasi-norms in  $B_{p,q}^s$  and  $F_{p,q}^s$ , respectively.

#### 5.1. Abel–Poisson Means

Let  $0 < \theta < \infty$ . The means

$$\mathfrak{M}_v^\theta f(x) = \sum_{k \in \mathbb{Z}_n} e^{-|k/v|^\theta} \hat{f}(k) e^{ikx}, \quad v = 1, 2, \dots, \tag{75}$$

are called the Abel–Poisson (or Abel–Cartwright) means.  $\theta = 1$  corresponds to the Abel and  $\theta = 2$  to the Gauss–Weierstrass mean.

LEMMA 2. *Let  $0 < \theta < \infty$ . Then the function  $\psi_\theta(\xi) = e^{-|\xi|^\theta}$  satisfies (20) and (21) with  $\sigma = \theta$  for all  $\lambda > 0$ . Furthermore,  $F^{-1}\psi_\theta \in L_1(R_n)$ .*

*Proof.* Let us put  $\eta(\xi) = |\xi|^{-\theta}(1 - \psi_\theta(\xi))$ . Obviously,  $\eta(\xi)$  is infinitely differentiable on  $R_n \setminus \{0\}$  and it holds that

$$D^\alpha \eta(\xi) = D^\alpha \left( \sum_{j=0}^{\infty} (-1)^{j+1} \frac{|\xi|^{\theta j}}{(j+1)!} \right) = |\xi|^{-|\alpha|} \eta_\alpha(\xi), \quad \xi \neq 0,$$

where  $\eta_\alpha(\xi)$  is bounded on  $\{\xi \mid 0 < |\xi| < \delta_1\}$  for all  $0 < \delta_1 < \infty$ . Hence, (39) is satisfied for  $\psi_\theta(\xi)$  for any  $\delta_1, 0 < \delta_1 < \infty$ , and any  $\lambda, 0 < \lambda < \infty$ . This implies (20). On the other hand it is easy to see that (40) is fulfilled for all  $\lambda, 0 < \lambda < \infty$ , and all  $\delta_2, 0 < \delta_2 < \infty$ . This shows the validity of (21). Now, let  $h(\xi)$  be the function from (22). We decompose  $\psi_\theta(\xi) = \psi_\theta(\xi) h(\xi) + \psi_\theta(\xi)(1 - h(\xi))$ . The function  $\psi_\theta(\xi)(1 - h(\xi))$  belongs to  $S(R_n)$ . Furthermore,

$$\psi_\theta(\xi) h(\xi) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} (|\xi|^\theta \tilde{h}(\xi))^j h(\xi), \tag{76}$$

where  $\tilde{h} \in S(R_n)$  such that  $\tilde{h}(\xi) = 1$  if  $\xi \in \text{supp } h$ . Using (35) one can show that  $|\xi|^\theta \tilde{h}(\xi) \in B_{1,\infty}^{n+\theta}(R_n)$  (cf. also [27, Remark 2.5.16/1]).  $B_{1,\infty}^{n+\theta}(R_n)$  is an algebra with respect to pointwise multiplication (cf. H. Triebel [26, Theorem 2.8.3]). Hence, (76) yields that  $\psi_\theta(\xi) h(\xi) \in B_{1,\infty}^{n+\theta}(R_n)$ . Thus, we have

$$|F^{-1}(\psi_\theta(\cdot) h(\cdot))(y)| \leq c(1 + |y|)^{-\lambda}$$

for all  $\lambda, 0 < \lambda < n + \theta$  (cf. Remark 3 and [16, Corollary 2]). Combining these results we obtain that  $F^{-1}\psi_\theta \in L_1(R)$ . This proves the lemma.

Lemma 2 shows that Theorems 1–4 can be applied to the means  $\mathfrak{B}_\nu^\theta f(x)$ ,  $\theta > 0$ . All the assumptions which we need are satisfied if we choose  $\sigma = \theta$  and  $\lambda > n$ . This yields the following theorem.

THEOREM 5. *Let  $M_\nu^\psi f(x) = \mathfrak{B}_\nu^\theta f(x)$ ,  $\nu = 1, 2, \dots, 0 < \theta < \infty$ .*

(i) *If  $0 < p \leq \infty, 0 < q \leq \infty$ , and  $n(1/\min(1, p) - 1) < s < \theta$  then (71) and (73) are equivalent quasi-norms in  $B_{p,q}^s$ .*

(ii) *If  $0 < u < \infty, 0 < u \leq p \leq \infty, 0 < q \leq \infty$ , and  $n(1/\min(1, u) - 1) < s < \theta$  then (72) is an equivalent quasi-norm in  $B_{p,q}^s$ .*

(iii) *If  $0 < p < \infty, 0 < q \leq \infty$ , and  $n(1/\min(1, p, q) - 1) < s < \theta$  then (74) is an equivalent quasi-norm in  $F_{p,q}^s$ .*

*Remark 9.* Let us note that the Corollary, in particular (68) and (69), holds true with  $\mathfrak{B}_v^\theta f$  instead of  $M_v^\psi f$ . This establishes a result concerning the strong summability of Abel–Poisson means. For the problem of strong summability and strong approximation we refer to L. Leindler [9].

*Remark 10.* It has been proved by B. I. Golubov [7] that  $f \in B_{p,\infty}^s$ ,  $0 < s \leq \theta$ ,  $1 \leq p \leq \infty$ , implies that (71), with  $q = \infty$ , is finite. Except for the limiting case  $s = \theta$  this coincides with one direction of Part (i) of the theorem. The corresponding one-dimensional result has been proved earlier, cf. P. L. Butzer and R. J. Nessel [3] and W. Trebels [25] and the literature cited there. Moreover, if

$$v^\theta \|f(x) - \mathfrak{B}_v^\theta f(x)\|_{L_p} = o(1), \quad (v \rightarrow \infty), \tag{77}$$

where  $1 \leq p \leq \infty$ , then  $f \equiv \text{const}$ . This can be found, for example in W. Trebels [25]. Hence, the upper bound  $\theta$ ,  $s \leq \theta$ , is quite natural. If  $s < n/(1/\min(1, p) - 1)$  then the assertions of the theorem cannot be true because the  $\delta$ -distribution belongs to  $B_{p,q}^s$  and  $F_{p,q}^s$  in this case. Thus some limiting cases remain open. Furthermore, the question arises whether the  $q$ -dependence in part (iii) can be removed.

### 5.2. Bessel-Potential Means

Let  $0 < \beta < \infty$ . The means

$$\Omega_v^\beta f(x) = \sum_{k \in \mathbb{Z}_n} \left(1 + \left|\frac{k}{v}\right|^2\right)^{-\beta/2} \hat{f}(k) e^{ikx}, \quad v = 1, 2, \dots, \tag{78}$$

are called the Bessel-potential means.

**LEMMA 3.** *Let  $0 < \beta < \infty$ . The function  $\psi_\beta(\xi) = (1 + |\xi|^2)^{-\beta/2}$  satisfies (20) and (21) with  $\sigma = 2$  for all  $\lambda$ ,  $0 < \lambda < \infty$ . Furthermore,  $F^{-1}\psi_\beta \in L_1(\mathbb{R}_n)$ .*

*Proof.* It follows from the asymptotic behavior of  $F^{-1}\psi_\beta$  that  $F^{-1}\psi_\beta \in L_1(\mathbb{R}_n)$ , cf., for example, S. M. Nikol'skij [13, formula 8.1/(6)]. The function  $|\xi|^{-2} (1 - (1 + |\xi|^2)^{-\beta/2})$  is, infinitely differentiable on  $\mathbb{R}_n$ . Hence, we have (39) with  $\sigma = 2$  for all  $\lambda$  and  $\delta_1$ . Moreover, straightforward calculations shows that (40) holds for all  $\lambda$  and  $\delta_2$ , too. By Remark 3 this proves the lemma.

**THEOREM 6.** *Let  $M_v^\psi f(x) = \Omega_v^\beta f(x)$ ,  $v = 1, 2, \dots$ ,  $0 < \beta < \infty$ .*

(i) *If  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $n/(1/\min(1, p) - 1) < s < 2$  then (71) and (73) are equivalent quasi-norms in  $B_{p,q}^s$ .*

(ii) *If  $0 < u < \infty$ ,  $0 < u \leq p \leq \infty$ ,  $0 < q \leq \infty$ , and  $n/(1/\min(1, u) - 1) < s < 2$  then (72) is an equivalent quasi-norm in  $B_{p,q}^s$ .*

(iii) If  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $n(1/\min(1, p, q) - 1) < s < 2$  then (74) is an equivalent quasi-norm in  $F_{p,q}^s$ .

*Remark 11.* Remark 10 applies to the means  $\mathfrak{Q}_v^\beta f$ , too. Furthermore, it can be found in W. Trebels [25] that

$$v^2 \|f(x) - \mathfrak{Q}_v^\beta f(x)\|_{L_p} = o(1) \quad (v \rightarrow \infty),$$

$1 \leq p \leq \infty$ ,  $0 < \beta < \infty$  implies that  $f \equiv \text{const}$ . Hence the upper bound  $s = 2$  is quite natural. For the lower bounds see Remark 10. The limiting cases are not covered by our theorem.

### 5.3. Riesz Means

Let  $0 < \alpha < \infty$ ,  $0 < \theta < \infty$ . The means

$$\mathfrak{R}_v^{\alpha,\theta} f(x) = \sum_{|k| \leq v} \left(1 - \left|\frac{k}{v}\right|^\theta\right)^\alpha \hat{f}(k) e^{ikx}, \quad v = 1, 2, \dots, \quad (79)$$

are called the Riesz means (Bochner–Riesz means if  $\theta = 2$ ).

**LEMMA 4.** Let  $0 < \alpha < \infty$ ,  $0 < \theta < \infty$ . The function  $\psi_{\alpha,\theta}(\xi) = (1 - |\xi|^\theta)_+^\alpha$  satisfies (20) and (21) with  $\sigma = \theta$  for all  $\lambda$ ,  $0 < \lambda < \alpha + (n + 1)/2$ . Furthermore,  $F^{-1}\psi_{\alpha,\theta} \in L_1(\mathbb{R}_n)$  if  $\alpha > (n - 1)/2$ .

*Proof.* Clearly, (40) holds for all  $\lambda$ ,  $0 < \lambda < \infty$  and all  $\delta_2$ ,  $\delta_2 > 1$ . Furthermore, it is known that  $\psi_{\alpha,\theta}(\xi)$  satisfies (38) with  $\lambda = \alpha + (n + 1)/2$ , cf. J. Peetre [14, p. 215].

This implies that  $F^{-1}\psi_{\alpha,\theta} \in L_1(\mathbb{R}_n)$  if  $\alpha > (n - 1)/2$ . By Remark 3 we obtain the lemma.

The lemma shows that we can apply our general results to the means  $\mathfrak{R}_v^{\alpha,\theta} f$  if  $\alpha > (n - 1)/2$ . This leads to the following theorem.

**THEOREM 7.** Let  $M_v^\psi f(x) = \mathfrak{R}_v^{\alpha,\theta} f(x)$ ,  $v = 1, 2, \dots$ ,  $0 < \theta < \infty$ , and  $\alpha > (n - 1)/2$ .

- (i) If  $2n/(2\alpha + n + 1) < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $n(1/\min(1, p) - 1) < s < \theta$  then (71) and (73) are equivalent quasi-norms in  $B_{p,q}^s$ .
- (ii) If  $2n/(2\alpha + n + 1) < p \leq \infty$ ,  $0 < u < \infty$ ,  $0 < u \leq p$ ,  $0 < q \leq \infty$ , and  $n(1/\min(1, u) - 1) < s < \theta$  then (72) is an equivalent quasi-norm in  $B_{p,q}^s$ .
- (iii) If  $2n/(2\alpha + n + 1) < p < \infty$ ,  $2n/(2\alpha + n + 1) < q \leq \infty$ , and  $n(1/\min(1, p, q) - 1) < s < \theta$  then (74) is an equivalent quasi-norm in  $F_{p,q}^s$ .

*Remark 12.* The above theorem extends our results obtained in [16, Theorem 8] to the case  $\theta \neq 2$ . Again, the case  $s = \theta$  is related to the saturation class of the means  $\mathfrak{R}_v^{\alpha,\theta} f$ , cf. W. Trebels [25] or P. L. Butzer and

R. J. Nessel [3, Chaps. 12, 13]. Moreover,  $\alpha = (n-1)/2$  corresponds to Bochner's critical index, cf. B. M. Stein and G. Weiss [24, Chapter 7]. The question arises whether Theorem 7 remains true for some  $\alpha \leq (n-1)/2$  if  $1 < p < \infty$ . Partial results concerning part (i) can be found in W. Trebels [25, 5.2] and the references given there and in [23, 6.1.3]. For characterizations by the norms (71) (with  $1 \leq p \leq \infty$ ) for Riesz means we refer also to J. Löfström [11], R. M. Trigub [29], P. L. Butzer and R. J. Nessel [3], and to the comparison theorems by H. S. Shapiro [18–20], and J. Boman and H. S. Shapiro [2].

*Remark 13.* Theorem 5 and Theorem 7 show that the Abel–Poisson means  $\mathfrak{B}_v^\theta f$  and the Riesz means  $\mathfrak{R}_v^{\alpha, \theta} f$  have the same approximation properties provided that  $\alpha > (n-1)/2$ ,  $1 \leq p \leq \infty$ ,  $1 \leq q \leq \infty$ . Concerning the characterization of the spaces  $B_{p,q}^s$  this is well known, cf. the comparison theorems mentioned in the preceding Remark. For values  $p, q$  less than 1 we have different properties; in particular, the admissible parameters  $p$  and  $q$  depend on  $\alpha$  ( $p > 2n/(2\alpha + n + 1)$ ,  $q > 2n/(2\alpha + n + 1)$ ) in the case of the Riesz means. This is caused by the smoothness properties of  $\psi_{\alpha, \theta}(\xi)$  at the points  $\xi$ ,  $|\xi| = 1$ . Furthermore, let us refer to the negative results concerning the pointwise comparison of Abel–Poisson and Riesz means obtained by W. Dickmeis, R. J. Nessel, and E. van Wickeren [4].

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